The Helmholtz equation in random media: well-posedness and a priori bounds*

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Abstract. We prove well-posedness results and a priori bounds on the solution of the Helmholtz equation $\nabla\cdot(A\nabla u)+k^2nu=-f$, posed either in $\mathbb{R}^d$ or in the exterior of a star-shaped Lipschitz obstacle, for a class of random A and n, random data f, and for all $k>0$. The particular class of A and n and the conditions on the obstacle ensure that the problem is nontrapping almost surely. These are the first well-posedness results and a priori bounds for the stochastic Helmholtz equation for arbitrarily large k and for A and n varying independently of k. These results are obtained by combining recent bounds on the Helmholtz equation for deterministic A and n and general arguments (i.e. not specific to the Helmholtz equation) presented in this paper for proving a priori bounds and well-posedness of variational formulations of linear elliptic stochastic PDEs. We emphasise that these general results do not rely on either the Lax-Milgram theorem or Fredholm theory, since neither are applicable to the stochastic variational formulation of the Helmholtz equation.

Key words. Helmholtz equation, random media, well-posedness, a priori bounds, high frequency, nontrapping

AMS subject classifications. 35J05, 35R60, 60H15

1. Introduction.

The goals of this paper are to prove results on the well-posedness of variational formulations of the stochastic Helmholtz equation

$$\nabla\cdot(A(\omega)\nabla u(\omega))+k^2n(\omega)u(\omega)=-f(\omega), \tag{1.1}$$

as well as a priori bounds on its solution that are explicit in the wavenumber k and the material coefficients A and n.

We consider (1.1) with physical domain either $\mathbb{R}^d$, $d=2,3$, or $\mathbb{R}^d\setminus\overline{D_-}$, where D_- (referred to as the *obstacle*) is a bounded, Lipschitz, open set such that $\mathbb{R}^d\setminus\overline{D_-}$ is connected, and

- ω is an element of the underlying probability space,
- A is a symmetric-positive-definite matrix-valued random field such that $\operatorname{ess\,supp}(I-A)$ is compact,
- n is a positive real-valued random field such that $\operatorname{ess\,supp}(1-n)$ is compact,
- f is a real-valued random field such that $\operatorname{ess\,supp} f$ is compact, and
- $k>0$ is the wavenumber,

and we are particularly interested in the case where the wavenumber k is large.

Motivation. The motivation for establishing well-posedness and proving a priori bounds on the solution of (1.1) is the growing interest in Uncertainty Quantification (UQ) for the Helmholtz equation; see e.g. [55, 51, 8, 22, 18, 19, 36, 30, 4]. (In this PDE context, by 'UQ' we mean theory and algorithms for computing statistics of quantities of interest involving

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PDEs *either* posed on a random domain *or* having random coefficients.) There is a large literature on UQ for the stationary diffusion equation

$$-\nabla \cdot (\kappa(\omega)\nabla u(\omega)) = f(\omega), \tag{1.2}$$

due in part to its large number of applications (e.g. in modelling groundwater flow), and a priori bounds on the solution are vital for the rigorous analysis of UQ algorithms; see e.g. [3, 2, 24, 41, 15]. In contrast, whilst (1.1) has many applications (e.g. in geophysics and electromagnetics), there is much less rigorous theory of UQ for the Helmholtz equation. The main reason for this is that the (deterministic) PDE theory of (1.1) when k is large is much more complicated that the analogous theory for (1.2).

Related previous work. To our knowledge, the only work that considers (1.1) with large k and attempts to establish either (i) well-posedness of variational formulations or (ii) a priori bounds is [18], which considers both (i) and (ii) for (1.1) posed in a bounded domain with an impedance boundary condition. We discuss the results of [18] further in subsection 1.3, but we highlight here that (a) [18] considers $A = I$ and $n = 1+\eta$, with η random and the magnitude of η decreasing with k, whereas we consider classes of A and n that allow k-independent random perturbations, and (b) in its well-posedness result, [18] invokes Fredholm theory to conclude existence of a solution, but this relies on an incorrect assumption about compact inclusion of Bochner spaces—see Appendix A below. In subsection 1.3 we also discuss the papers [8, 31, 32, 30] on the theory of UQ for either (1.1) or the related time-harmonic Maxwell's equations; in these papers either the k-explicit well-posedness is not a primary concern or k is assumed to be small. Our hope is that the results in the present paper can be used in the rigorous theory of UQ for Helmholtz problems with large k.

The contributions of this paper. The main results in this paper, Theorems 1.4 and 1.8 below, concern well-posedness and a priori bounds for the solutions of various formulations of the stochastic Helmholtz equation; these formulations include those used in sampling-based UQ algorithms (Problems 1 and 2 below) and in the stochastic Galerkin method (Problem 3 below). These are the first such results for arbitrarily large k and for A and n varying independently of k. These results are proved by combining:

1. bounds for the Helmholtz equation in [25] with A and n deterministic but spatially-varying, with
2. general arguments (i.e. not specific to Helmholtz) presented here for proving a priori bounds and well-posedness of variational formulations of linear elliptic SPDEs.

Regarding 1: the k-dependence of the bounds on u in terms of f depends crucially on whether or not A, n, and D_- are such that there exist trapped rays. In the trapping case, the solution operator can grow exponentially in k (see [46, 9, 45, 11, 5] and [6, Section 2.5], and the reviews in [40, Section 6], [13, Section 1.1], and [25, Section 1]); in contrast, in the nontrapping case, the solution operator is bounded uniformly in k (see [52, 10] and the references therein). The bounds in [25] are under conditions on A, n, and D_- that ensure nontrapping of rays; the significance of these bounds is that they are the first (deterministic) bounds for the Helmholtz scattering problem in which both A and n vary and the bounds are explicit in A and n (as well as in k). This feature of being explicit in A and n is crucial in allowing us to prove the results in this paper when A and n are random fields.

Regarding 2: the main reason these general arguments are needed is the fact that the variational formulations of both the deterministic and the stochastic Helmholtz equation are not coercive, and so one cannot use the Lax–Milgram theorem to conclude well-posedness and an a priori bound. In the deterministic case, the remedy for the lack of coercivity of the Helmholtz equation is to use Fredholm theory, but this is *not* applicable to the stochastic variational formulation of the Helmholtz equation because the necessary compactness results do not hold in Bochner spaces (see Appendix A below). Our solution to this lack of coercivity and failure of Fredholm theory is to use well-posedness results and bounds from the deterministic case to prove results for the stochastic case. We work 'pathwise' by integrating the deterministic results over probability space and identifying conditions under which the necessary quantities are indeed integrable. Our approach is given in a general framework that, given (i) deterministic well-posedness results and a priori bounds that are explicit in all the coefficients, and (ii) measurability and integrability conditions on the stochastic quantities, returns corresponding well-posedness results, a priori bounds, and equivalence results for different formulations of the stochastic problem. One reason we state our well-posedness results in general (i.e. not only in the specific case of the Helmholtz equation) is that we expect that they can be used in the future to prove well-posedness results for the time-harmonic Maxwell's equations in random media. A nontechnical summary of the ideas behind our well-posedness results is given in Remark 2.12 below. Some of these results are similar in spirit to the results about the PDE (1.2) in [24, 41] (which deal with the failure of Lax–Milgram for the stochastic variational problem for (1.2) in the case when the coefficient κ is not uniformly bounded above and below), and our arguments use some of the ideas and technical tools from these two papers.

1.1. Statement of main results.

Notation and basic definitions. Let either (i) $D_- \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz open set such that $\mathbf{0} \in D_-$ and the open complement $D_+ := \mathbb{R}^d \setminus \overline{D_-}$ is connected, or (ii) $D_- = \emptyset$. Let $\Gamma_D = \partial D_-$. Fix $R > 0$ and let B_R be the ball of radius R centred at the origin. Define $\Gamma_R := \partial B_R$ and $D_R := D_+ \cap B_R$ (see Figure 1.1). Let γ denote the trace operator from D_R to $\partial D_R = \Gamma_D \cup \Gamma_R$ and define $H^1_{0,D}(D_R) := \big\{v \in H^1(D_R) : \gamma v = 0 \text{ on } \Gamma_D\big\}$.

Let $T_R : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$ be the Dirichlet-to-Neumann map for the deterministic equation $\Delta u + k^2 u = 0$ posed in the exterior of B_R with the Sommerfeld radiation condition

$$\frac{\partial u}{\partial r}(\mathbf{x}) - \mathrm{i}ku(\mathbf{x}) = o\left(\frac{1}{r^{(d-1)/2}}\right) \text{ as } r := |\mathbf{x}| \to \infty, \text{ uniformly in } \frac{\mathbf{x}}{|\mathbf{x}|}; \tag{1.3}$$

see [42, Section 2.6.3] and [12, Equations 3.5 and 3.6] for an explicit expression for T_R in terms of Hankel functions and Fourier series ($d = 2$)/spherical harmonics ($d = 3$). Let $\langle\cdot,\cdot\rangle_{\Gamma_R}$ be the duality pairing on Γ_R between $H^{-1/2}(\Gamma_R)$ and $H^{1/2}(\Gamma_R)$ and write $\mathrm{d}\lambda$ for Lebesgue measure.

Let $L^\infty\big(D_+;\mathbb{R}^{d\times d}\big)$ be the set of all matrix-valued functions $A : D_+ \to \mathbb{R}^{d\times d}$ such that $A_{i,j} \in L^\infty(D_+;\mathbb{R})$ for all $i, j = 1, \ldots, d$. Where the range of functions is $\mathbb{C}$ we suppress the second argument in a function space, e.g. we write $L^\infty(D_+)$ for $L^\infty(D_+;\mathbb{C})$. We write $D_1 \subset\subset D_2$ if D_1 is a compact subset of the open set D_2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Throughout this paper, unless stated otherwise we equip a topological space with its Borel σ-algebra. See Appendix B for a summary of the measure-theoretic concepts used in this paper. Let

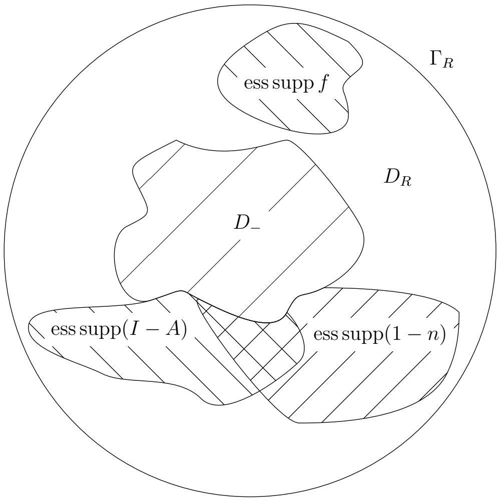

Figure 1.1. *Examples of the domains D_- and D_R, the set Γ_R, and essential supports of $I-A$, $1-n$ and f in the definition of the Helmholtz stochastic EDP.*

- $f : \Omega \to L^2(D_+)$ be such that $\operatorname{ess\,supp} f \subset\subset B_R$ almost surely
- $n : \Omega \to L^\infty(D_+; \mathbb{R})$ be such that $\operatorname{ess\,supp}(1-n) \subset\subset B_R$ almost surely and there exist $n_{\min}, n_{\max} : \Omega \to \mathbb{R}$ such that $0 < n_{\min}(\omega) \leq n(\omega)(\mathbf{x}) \leq n_{\max}(\omega)$ for almost every $\mathbf{x} \in D_+$ almost surely, and
- $A : \Omega \to L^\infty\big(D_+; \mathbb{R}^{d\times d}\big)$ be such that $\operatorname{ess\,supp}(I-A) \subset\subset B_R$, $A_{ij} = A_{ji}$ almost surely, and there exist $A_{\min}, A_{\max} : \Omega \to \mathbb{R}$ such that $0 < A_{\min}(\omega) < A_{\max}(\omega)$ almost surely and $A_{\min}(\omega)|\boldsymbol{\xi}|^2 \leq \big(A(\omega)(\mathbf{x})\boldsymbol{\xi}\big) \cdot \boldsymbol{\xi} \leq A_{\max}(\omega)|\boldsymbol{\xi}|^2$ for almost every $\mathbf{x} \in D_+$ and for all $\boldsymbol{\xi} \in \mathbb{C}^d$ almost surely.

If $v : \Omega \to Z$ for some function space Z of functions on $\mathbb{R}^d$, we abuse notation slightly and write $v(\omega, \mathbf{x})$ instead of $v(\omega)(\mathbf{x})$.

Variational Formulations. We consider three different formulations of the *Helmholtz stochastic exterior Dirichlet problem* (stochastic EDP); Problems 1–3 below.

Define the sesquilinear form $a(\omega)$ on $H^1_{0,D}(D_R) \times H^1_{0,D}(D_R)$ by

$$[a(\omega)](v_1, v_2) \coloneqq \int_{D_R} \Big((A(\omega)\nabla v_1) \cdot \nabla \overline{v_2} - k^2 n(\omega)\, v_1\, \overline{v_2} \Big)\, \mathrm{d}\lambda - \big\langle T_R \gamma v_1, \gamma v_2 \big\rangle_{\Gamma_R}, \tag{1.4}$$

and the antilinear functional $L(\omega)$ on $H^1_{0,D}(D_R)$ by

$$[L(\omega)](v_2) \coloneqq \int_{D_R} f(\omega)\, \overline{v_2}\, \mathrm{d}\lambda. \tag{1.5}$$

Define the sesquilinear form $\mathfrak{a}$ on $L^2\big(\Omega; H^1_{0,D}(D_R)\big) \times L^2\big(\Omega; H^1_{0,D}(D_R)\big)$ and the antilinear functional $\mathfrak{L}$ on $L^2\big(\Omega; H^1_{0,D}(D_R)\big)$ by

$$\mathfrak{a}(v_1, v_2) \coloneqq \int_\Omega [a(\omega)](v_1(\omega), v_2(\omega))\, \mathrm{d}\mathbb{P}(\omega) \quad \text{and} \quad \mathfrak{L}(v_2) \coloneqq \int_\Omega [L(\omega)](v_2(\omega))\, \mathrm{d}\mathbb{P}(\omega). \tag{1.6}$$

We consider the following three problems:

Problem 1 (Measurable EDP almost surely). *Find a measurable* $u : \Omega \to H^1_{0,D}(D_R)$ *such that*

$$[a(\omega)](u(\omega), v) = [L(\omega)](v) \text{ for all } v \in H^1_{0,D}(D_R) \text{ almost surely.}$$

Problem 2 (Second-order EDP almost surely). *Find* $u \in L^2\big(\Omega; H^1_{0,D}(D_R)\big)$ *such that*

$$[a(\omega)](u(\omega), v) = [L(\omega)](v) \text{ for all } v \in H^1_{0,D}(D_R) \text{ almost surely.}$$

Problem 3 (Stochastic variational EDP). *Find* $u \in L^2\big(\Omega; H^1_{0,D}(D_R)\big)$ *such that*

$$\mathfrak{a}(u, v) = \mathfrak{L}(v) \text{ for all } v \in L^2\big(\Omega; H^1_{0,D}(D_R)\big).$$

Problem 2 is the foundation of sampling-based UQ methods, such as Monte-Carlo and Stochastic-Collocation methods; its analogue for the stationary diffusion equation is well-studied in, e.g., [54, 2, 43, 14, 15, 50, 35, 29]. Similarly Problem 3 is the foundation of the Stochastic Galerkin method (a finite element method in $\Omega \times D$, where D is the spatial domain), and is studied for the Helmholtz Interior Impedance Problem in [18], and its analogue for the stationary diffusion equation is considered in, e.g., [3, 34, 27].

Remark 1.1 (Why consider Problem 1?).
The difference between Problems 1 and 2 is that Problem 1 requires no integrability of u over Ω, whereas Problem 2 requires $u \in L^2(\Omega, H^1_{0,D}(D_R))$. Since all the theory for sampling-based UQ methods assume some integrability of the solution, the natural question is: why consider Problem 1 at all? The main reason we consider Problem 1 is that, given the existing PDE theory for the Helmholtz equation, we can prove existence of a solution to Problem 1 under general conditions on A and n, but there is no current prospect of proving existence of a solution to Problem 2 under general conditions on A and n. The explanation for this consists of the following three points:

1. The only two known ways to obtain a solution to Problem 2 are: (i) obtain a deterministic a priori bound, explicit in all parameters, and integrate (followed, e.g., in [15] for (1.2) with lognormal coefficients) and (ii) obtain a solution to Problem 3 and show this is a solution to Problem 2. In the Helmholtz case, doing (ii) is difficult as neither the Lax–Milgram theorem nor Fredholm theory is applicable (as explained in the introduction), and so we follow the approach in (i).
2. The only known bounds on the solution of the Helmholtz equation explicit in all parameters are those recently obtained for nontrapping scenarios in [25, 21].
3. Obtaining a bound explicit in all parameters for a general class of A and n, e.g., $A \in W^{1,\infty}\big(D_R; \mathbb{R}^{d\times d}\big)$ and $n \in L^\infty(D_R; \mathbb{R})$ is well beyond current techniques. Indeed, a general class of A and n will include both trapping and nontrapping scenarios, and such a bound would need to capture the exponential blow-up in k for trapping A and n, the uniform boundedness in k for nontrapping A and n, and be explicit in A and n.

Given this fact that there is no current prospect of proving existence of a solution to Problem 2 under general conditions on A and n we keep Problem 1 so that we prove an (albeit weaker) existence result for the Helmholtz equation with general coefficients.

Remark 1.2 (Measurability of u in Problem 1). It is natural to construct the solution of Problem 1 pathwise; that is, one defines $u(\omega)$ to be the solution of the deterministic problem with coefficients $A(\omega)$ and $n(\omega)$. However, is it then not obvious that u is measurable. In the proof of Theorem 1.4 below, we show that the measurability of u follows from (i) a natural condition on the measurability of the coefficients and data (Condition C1 below), and (ii) the continuity of the map taking the coefficients of the deterministic PDE to the solution of the deterministic PDE (see Lemma 4.12 below).

In Theorems 1.4 and 1.8 we prove results on the well-posedness of Problems 1–3 under conditions on A, n, f, and D_-. Although A, n, and f are defined on D_+, since $\operatorname{ess\,supp}(I-A)$, $\operatorname{ess\,supp}(1-n)$, and $\operatorname{ess\,supp} f$ are compactly contained in D_R we can consider A, n, and f as functions on D_R.

Condition 1.3 (Regularity and stochastic regularity of f, A, and n). The random fields f, A, and n satisfy $f \in L^2\big(\Omega; L^2(D_R)\big)$, $A : \Omega \to W^{1,\infty}\big(D_R; \mathbb{R}^{d\times d}\big)$ with $A \in L^\infty\big(\Omega; L^\infty\big(D_R; \mathbb{R}^{d\times d}\big)\big)$, and $n \in L^\infty(\Omega; L^\infty(D_R; \mathbb{R}))$.

Theorem 1.4 (Equivalence of variational problems). *Under Condition 1.3:*

- *The maps $\mathfrak{a}$ and $\mathfrak{L}$ (defined by (1.6)) are well-defined.*
- *$u \in L^2\big(\Omega; H^1_{0,D}(D_R)\big)$ solves Problem 2 if and only if u solves Problem 3.*
- *If $u \in L^2\big(\Omega; H^1_{0,D}(D_R)\big)$ solves Problem 2, then any member of the equivalence class of u solves Problem 1.*
- *The solution of Problem 1 exists and is unique up to modification on a set of measure zero in Ω.*
- *The solution of Problems 2 and 3 is unique in $L^2\big(\Omega; H^1_{0,D}(D_R)\big)$.*

Observe that the only relationship between formulations not proved in Theorem 1.4 is: if $u : \Omega \to H^1_{0,D}(D_R)$ solves Problem 1 then $u \in L^2\big(\Omega; H^1_{0,D}(D_R)\big)$ and u solves Problem 2. Theorem 1.8 below includes this relationship, under additional assumptions on A, n, and D_-.

Definition 1.5 (A particular class of (deterministic) nontrapping coefficients). *Let $\mu_1, \mu_2 > 0$, $A_0 \in W^{1,\infty}\big(D_R; \mathbb{R}^{d\times d}\big)$ with $\operatorname{ess\,supp}(I-A_0) \subset\subset B_R$, and $n_0 \in W^{1,\infty}(D_R; \mathbb{R})$ with $\operatorname{ess\,supp}(1-n_0) \subset\subset B_R$. We write $A_0 \in \mathrm{NT}_A(\mu_1)$ and $n_0 \in \mathrm{NT}_n(\mu_2)$ if*

$$A_0(\mathbf{x}) - (\mathbf{x}\cdot\nabla)A_0(\mathbf{x}) \geq \mu_1 \quad \text{and} \quad n_0(\mathbf{x}) + \mathbf{x}\cdot\nabla n_0(\mathbf{x}) \geq \mu_2 \tag{1.7}$$

for almost every $\mathbf{x} \in D_R$, where the first inequality holds in the sense of quadratic forms.

Condition 1.6 (k-independent nontrapping conditions on (random) A and n). The random fields A and n satisfy $A : \Omega \to W^{1,\infty}\big(D_R; \mathbb{R}^{d\times d}\big)$ and $n : \Omega \to W^{1,\infty}(D_R; \mathbb{R})$. Furthermore, there exist $\mu_1, \mu_2 : \Omega \to \mathbb{R}$, independent of f, with $\mu_1(\omega), \mu_2(\omega) > 0$ almost surely and $1/\mu_1, 1/\mu_2 \in L^2(\Omega; \mathbb{R})$ such that $A(\omega) \in \mathrm{NT}_A(\mu_1(\omega))$ almost surely and $n(\omega) \in \mathrm{NT}_n(\mu_2(\omega))$ almost surely.

Definition 1.7 (Star-shaped). *The set $D \subseteq \mathbb{R}^d$ is* star-shaped with respect to *the point $\mathbf{x_0}$ if for any $\mathbf{x} \in D$ the line segment $[\mathbf{x_0}, \mathbf{x}] \subseteq D$.*

Theorem 1.8 (Equivalence of variational problems in a nontrapping case). *Let D_- be star-shaped with respect to the origin. Under Conditions 1.3 and 1.6:*

- *The maps $\mathfrak{a}$ and $\mathfrak{L}$ (defined by (1.6)) are well-defined.*

- *Problems 1–3 are all equivalent.*
- *The solution $u \in L^2\big(\Omega; H^1_{0,D}(D_R)\big)$ of these problems exists, is unique, and, given $k_0 > 0$, satisfies the bound*

$$\|\nabla u\|^2_{L^2(\Omega;L^2(D_R))} + k^2\|u\|^2_{L^2(\Omega;L^2(D_R))} \leq \|C_1\|_{L^1(\Omega)}\|f\|^2_{L^2(\Omega;L^2(D_R))} \tag{1.8}$$

for all $k \geq k_0$, where $C_1 : \Omega \to \mathbb{R}$ is given by

$$C_1 = \max\left\{\frac{1}{\mu_1}, \frac{1}{\mu_2}\right\}\left(\frac{R^2}{\mu_1} + \frac{2}{\mu_2}\left(R + \frac{d-1}{2k_0}\right)^2\right). \tag{1.9}$$

As highlighted above, Theorem 1.8 is obtained from combining deterministic a priori bounds from [25] with the general arguments in section 2 about well-posedness of variational formulations of stochastic PDEs. Theorem 1.8 uses the most basic a priori bound proved in [25] (from [25, Theorem 2.5]), but [25] contains several extensions of this bound. Remarks 1.9, 1.10, and 1.12–1.14 outline the implications that these (deterministic) extensions have for the stochastic Helmholtz equation.

Remark 1.9 (Dirichlet boundary conditions on Γ_D and plane-wave incidence). The formulations of the stochastic EDP above assume that $u = 0$ on the boundary Γ_D. An important scattering problem for which $u \neq 0$ on Γ_D is when u is the field scattered by an incident plane wave; in this case $\gamma u = -\gamma u_I$, where u_I is the incident plane wave. The results in this paper can be easily extended to the case when $u \neq 0$ on Γ_D using [25, Theorem 2.19(ii)] which proves a priori (deterministic) bounds in this case. One subtlety, however, is that f is then not necessarily independent of μ_1 and μ_2, indeed in this case $f = -\nabla \cdot (A\nabla u_I) - k^2 n u_I$. One can produce an analogue of Theorem 1.8 in the case where f, μ_1, and μ_2 are dependent, but one requires $1/\mu_1, 1/\mu_2 \in L^4(\Omega)$ and $f \in L^4\big(\Omega; L^2(D)\big)$; see Remark 4.17 below.

Remark 1.10 (The case when either $n = 1$ or $A = I$). When either $n = 1$ or $A = I$, [25, Theorem 2.19] gives deterministic bounds under weaker conditions on A and n respectively; the corresponding results for the stochastic case are that: When $n = 1$ almost surely, the condition $A(\omega) \in \mathrm{NT}_A(\mu_1(\omega))$ in Condition 1.6 can be improved to $2A(\omega) - (\mathbf{x}\cdot\nabla)A(\omega) \geq \mu_1(\omega)$ for almost every $\mathbf{x} \in D_+$, almost surely. When $A = I$ almost surely, the condition $n(\omega) \in \mathrm{NT}_n(\mu_2(\omega))$ in Condition 1.6 can be improved to: $2n(\omega) + \mathbf{x}\cdot\nabla n(\omega) \geq \mu_2(\omega)$ for almost every $\mathbf{x} \in D_+$, almost surely.

Remark 1.11 (Geometric interpretation of the conditions on A and n in Definition 1.5). Recall that the $k \to \infty$ asymptotics of solutions of the Helmholtz equation are governed by the behaviour of rays (see, e.g., [1]). The Helmholtz EDP is *nontrapping* if all rays starting in D_R escape from D_R after some uniform time (see, e.g., [10, Definition 1.1]); the EDP is *trapping* otherwise. The k-dependence of the solution operator depends strongly on whether the problem is trapping, and the type of trapping present; see, e.g., the overview discussions in [25, Section 1], [13, Section 1.1]. The conditions on A and n in Condition 1.6 and the star-shapedness restriction on D_- are sufficient for the Helmholtz stochastic EDP to be nontrapping almost surely. For more details on how these conditions are related to trapping, see [25, Theorem 7.7].

Remark 1.12 (The Helmholtz stochastic truncated exterior Dirichlet problem). It is common to approximate the Dirichlet-to-Neumann map on Γ_R, i.e. T_R, by an 'absorbing boundary condition', the simplest of which is the so-called impedance boundary condition. We call the Helmholtz stochastic EDP posed in D_R with an impedance boundary condition on Γ_R the stochastic *truncated exterior Dirichlet problem* (stochastic TEDP). The results in this paper also hold for the stochastic TEDP (with arbitrary Lipschitz truncation boundary) under an analogue of Condition 1.6 based on the deterministic bounds in [25, Theorem A.6(i)] instead of [25, Theorem 2.5].

Remark 1.13 (Discontinuous A and n). The requirements on A and n in Definition 1.5 require A and n to be continuous. In addition to proving deterministic a priori bounds for the class of A and n in Definition 1.5, the paper [25] also proves deterministic bounds for discontinuous A and n satisfying (1.7) in a distributional sense; see [25, Theorem 2.7]. The well-posedness results and a priori bounds in this paper can therefore be adapted to prove results about the stochastic Helmholtz equation for a class of random A and n that allows nontrapping jumps on randomly-placed star-shaped interfaces.

Remark 1.14 (k-dependent A and n). In this paper we focus on random fields A and n varying independently of k; this corresponds to a fixed physical medium, characterised by A and n, with waves of frequency k passing through. In subsection 1.2 below we construct A and n as (k-independent) $W^{1,\infty}$ perturbations of random fields A_0 and n_0 satisfying Condition 1.6. We note, however, that results for A and n being k-*dependent* L^∞ perturbations (i.e. rougher, but k-dependent perturbations) of A_0 and n_0 satisfying Condition 1.6 can easily be obtained.

The basis for these bounds is observing that *deterministic* a priori bounds hold when (a) $A \in \mathrm{NT}_A(\mu_1)$, $n = n_0 + \eta$, where $n_0 \in \mathrm{NT}_n(\mu_2)$ and $k\|\eta\|_{L^\infty(D_R;\mathbb{R})}$ is sufficiently small, and (b) $A = A_0 + B$, $n = n_0 + \eta$, where $A_0 \in \mathrm{NT}_A(\mu_1)$, $n_0 \in \mathrm{NT}_n(\mu_2)$, $k\|\eta\|_{L^\infty(D_R;\mathbb{R})}$ and $k\|B\|_{W^{1,\infty}(D_R;\mathbb{R}^{d\times d})}$ are both sufficiently small, and A, n, and D_- are such that $u \in H^2(D_R)$ (see, e.g., [39, Theorem 4.18(i)] for these latter requirements). Given these deterministic bounds, the general arguments in this paper can then be used to prove well-posedness of the analogous stochastic problems.

To understand why bounds hold in the case (a), observe that one can write the PDE as

$$\nabla\cdot(A\nabla u) + k^2 n_0 u = -f - k^2\eta u; \tag{1.10}$$

if $k\|\eta\|_{L^\infty(D_R;\mathbb{R})}$ is sufficiently small then the contribution from the $k^2\eta u$ term on the right-hand side of (1.10) can be absorbed into the $k^2\|u\|^2_{L^2(D_R)}$ term appearing on the left-hand side of the bound (the deterministic analogue of (1.8)). In the case $n_0 = 1$, this is essentially the argument used to prove the a priori bound in [18, Theorem 2.4] (see [25, Remark 2.15]). The reason bounds hold in the case (b) is similar, except now we need the H^2 norm of u on the left-hand side of the bound (as well as the H^1 norm) to absorb the contribution from the $\nabla\cdot(B\nabla u)$ term on the right-hand side.

1.2. Random fields satisfying Condition 1.6. The main focus of this paper is proving well-posedness of the variational formulations of the stochastic Helmholtz equation, and a priori bounds on the solution, for the most-general class of A and n allowed by the deterministic bounds in [25]. However, in this section, motivated by the Karhunen-Loève expansion (see e.g. [38, p. 201ff.]) and similar expansions of material coefficients for the stationary diffusion equation [35, Section 2.1], we consider A and n as series expansions around known non-random fields A_0 and n_0 satisfying Condition 1.6 (i.e., Condition 1.6 is satisfied for n_0, A_0 independent of $\omega \in \Omega$, and therefore μ_1, μ_2 independent of ω). Define

$$A(\omega,\mathbf{x}) = A_0(\mathbf{x}) + \sum_{j=1}^{\infty} Y_j(\omega)\Psi_j(\mathbf{x}) \quad \text{and} \quad n(\omega,\mathbf{x}) = n_0(\mathbf{x}) + \sum_{j=1}^{\infty} Z_j(\omega)\psi_j(\mathbf{x}), \tag{1.11}$$

where:

- $\operatorname{ess\,supp}(1 - A_0)$, $\operatorname{ess\,supp}(I - n_0) \subset\subset B_R$,
- A_0 and n_0 satisfy Condition 1.6 with μ_1 and μ_2 independent of $\omega \in \Omega$
- $Y_j, Z_j \sim \mathrm{Unif}(-1/2, 1/2)$ i.i.d.,
- $\Psi_j \in W^{1,\infty}\big(D_R; \mathbb{R}^{d\times d}\big)$ with $\operatorname{ess\,supp} \Psi_j \subset\subset B_R$ for all $j = 1, \ldots, m$,

  $$\sum_{j=1}^{\infty} \operatorname{ess\,sup}_{\mathbf{x}\in D_R} \|\Psi_j\|_2 < 2A_{0,\min} \quad \text{and} \quad \sum_{j=1}^{\infty} \|\Psi_j\|_{W^{1,\infty}(D_R;\mathbb{R}^{d\times d})} < \infty, \tag{1.12}$$

  where $A_{0,\min} > 0$ is such that $A_{0,\min}|\boldsymbol{\xi}|^2 \le \big(A(\mathbf{x})\boldsymbol{\xi}\big) \cdot \boldsymbol{\xi}$ for almost every $\mathbf{x} \in D_+$ and for all $\boldsymbol{\xi} \in \mathbb{C}^d$, and where $\|\cdot\|_2$ is the operator norm induced by the Euclidean vector norm on $\mathbb{C}^d$ (i.e., $\|\cdot\|_2$ is the spectral norm).
- $\psi_j \in W^{1,\infty}(D_R; \mathbb{R})$ with $\operatorname{ess\,supp} \psi_j \subset\subset B_R$ for all $j = 1, \ldots, m$,

  $$\sum_{j=1}^{\infty} \|\psi_j\|_{L^\infty(D_R;\mathbb{R})} < 2n_{0,\min} \quad \text{and} \quad \sum_{j=1}^{\infty} \|\psi_j\|_{W^{1,\infty}(D_R;\mathbb{R})} < \infty, \tag{1.13}$$

  where $n_{0,\min} := \operatorname{ess\,inf}_{\mathbf{x}\in D_R} n_0(\mathbf{x})$, and

The first assumptions in (1.12) and (1.13) ensure that $A > 0$ (in the sense of quadratic forms) and $n > 0$ almost surely, respectively. The second assumptions in (1.12) and (1.13) are used to prove A and n are measurable, respectively; see [44, Appendix C]. The following lemmas give sufficient conditions for the series in (1.11) to satisfy Condition 1.6.

Lemma 1.15 (Series expansion of A satisfies Condition 1.6). *Let $\mu > 0$, $\delta \in (0,1)$. If $A_0 \in \mathrm{NT}_A(\mu)$, and*

$$\sum_{j=1}^{\infty} \operatorname{ess\,sup}_{\mathbf{x}\in D_R} \|\Psi_j(\mathbf{x}) - (\mathbf{x}\cdot\nabla)\Psi_j(\mathbf{x})\|_2 \le 2\delta\mu, \tag{1.14}$$

then $A \in \mathrm{NT}_A((1-\delta)\mu)$ almost surely.

Proof of Lemma 1.15. Since $A_0 \in \mathrm{NT}_A(\mu)$, we have

$$\Big((A(\omega,\mathbf{x}) - (\mathbf{x}\cdot\nabla)A(\omega,\mathbf{x}))\boldsymbol{\xi}\Big)\cdot\overline{\boldsymbol{\xi}} \ge \mu|\boldsymbol{\xi}|^2 + \sum_{j=1}^{\infty}\Big(Y_j(\omega)(\Psi_j(\mathbf{x}) - (\mathbf{x}\cdot\nabla)\Psi_j(\mathbf{x}))\boldsymbol{\xi}\Big)\cdot\overline{\boldsymbol{\xi}} \tag{1.15}$$

for all $\boldsymbol{\xi} \in \mathbb{C}^d$, for almost every $\mathbf{x} \in D_R$, almost surely. As $Y_j \sim \mathrm{Unif}(-1/2, 1/2)$ for all j and the bound (1.14) holds, the right-hand side of (1.15) is bounded below by $(1-\delta)\mu|\boldsymbol{\xi}|^2$ almost surely. Since $\boldsymbol{\xi} \in \mathbb{C}^d$ was arbitrary, it follows that $A(\omega) \in \mathrm{NT}_A((1-\delta)\mu))$ almost surely, as required. ∎

Lemma 1.16 (Series expansion of n satisfies Condition 1.6). *Let $\mu > 0$ and $\delta \in (0,1)$. If $n_0 \in \mathrm{NT}_n(\mu)$ and $\sum_{j=1}^{m} \|\psi_j(\mathbf{x}) + \mathbf{x}\cdot\nabla\psi_j(\mathbf{x})\|_{L^\infty(D_R;\mathbb{R})} \leq 2\delta\mu$, then $n \in \mathrm{NT}_n((1-\delta)\mu)$.*

The proof of Lemma 1.16 is omitted, since it is similar to the proof of Lemma 1.15; in fact it is simpler, because it involves scalars rather than matrices.

1.3. Discussion of the main results in the context of other work on UQ for time-harmonic wave equations. In this section we discuss existing results on well-posedness of (1.1), as well as analogous results for the elastic wave equation and the time-harmonic Maxwell's equations. The most closely-related work to the current paper is [18] (and its analogue for elastic waves [20]), in that a large component of [18] consists of attempting to prove well-posedness and a priori bounds for the stochastic variational formulation (i.e. Problem 3) of the Helmholtz Interior Impedance Problem; i.e., (1.1) with $A = I$ and stochastic n posed in a bounded domain with an impedance boundary condition $\partial u/\partial\nu - iku = g$ (see the discussion of such boundary-value problems in Remark 1.12). Under the assumption of existence, [18] shows that for any $k > 0$ the solution is unique and satisfies an a priori bound of the form (1.8) (with different constant C_1), provided $n = 1 + \eta$ where the random field η satisfies (almost surely) $\|\eta\|_{L^\infty} \leq C/k$ for some $C > 0$ independent of k. [18] then invokes Fredholm theory to conclude existence, but this relies on an incorrect assumption about compact inclusion of Bochner spaces—see Appendix A below. However, combining Theorem 1.4 and Remarks 1.12 and 1.14 with $A = I$ and $n_0 = 1 + \eta$ (with η as above) produces an analogous result to Theorem 1.8, and gives a correct proof of [18, Theorem 2.5]. Therefore the analysis of the Monte Carlo interior penalty discontinuous Galerkin method in [18] can proceed under the assumptions of Theorem 1.4 and Remarks 1.12 and 1.14.

The paper [30] considers the Helmholtz transmission problem with a stochastic interface, i.e. (1.1) posed in $\mathbb{R}^d$ with both A and n piecewise constant and jumping on a common, randomly-located interface. A component of this work is establishing well-posedness of Problem 1 for this setup. To do this, the authors make the assumption that k is small (to avoid problems with trapping mentioned above—see the comments after [30, Theorem 4.3]); the sesquilinear form a is then coercive and an a priori bound (in principle explicit in A and n) follows [30, Lemma 4.5]. By Remark 1.13, the results of this paper can be used to obtain the analogous well-posedness result for large k in the case of nontrapping jumps.

The paper [8] studies the *Bayesian inverse problem* associated to (1.1) with $A = I$ and $n = 1$ posed in the exterior of a Dirichlet obstacle with random boundary. A component of the analysis in [8] is the well-posedness of the forward problem for an obstacle with a variable boundary [8, Proposition 3.5]. Instead of mapping the problem to one with a fixed domain and variable A and n, [8] instead works with the variability of the obstacle directly, using boundary-integral equations. The k-dependence of the solution operator is not considered, but would enter in [8, Lemma 3.1].

The papers [32] and [31] consider the time-harmonic Maxwell's equations with (i) the material coefficients ε, μ constant in the exterior of a perfectly-conducting random obstacle and (ii) ε, μ piecewise constant and jumping on a common randomly located interface; in both cases these problems are mapped to problems where the domain/interface is fixed and ε and μ are random and heterogeneous. The papers [32] and [31] essentially consider the analogue of Problem 1 for the time-harmonic Maxwell's equations, obtaining well-posedness from the corresponding results for the related deterministic problems.

1.4. Outline of the paper. In subsection 1.3 we discuss our results in the context of related literature. In section 2 we state general results on a priori bounds and well-posedness for stochastic variational formulations. In section 3 we prove the results in section 2. In section 4 we prove Theorems 1.4 and 1.8. In Appendix A we discuss the failure of Fredholm theory for the stochastic variational formulation of Helmholtz problems. In Appendix B we recap results from measure theory and the theory of Bochner spaces.

2. General results on proving a priori bounds and well-posedness of stochastic variational formulations.

In this section we state general results for proving a priori bounds and well-posedness results for variational formulations of linear elliptic SPDEs.

2.1. Notation and definitions of the variational formulations. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let X and Y be separable Banach spaces over a field $\mathbb{F}$, (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Let $\mathrm{B}(X, Y^*)$ denote the space of bounded linear maps $X \to Y^*$. Let $\mathcal{C}$ be a topological space with topology $\mathcal{T}_{\mathcal{C}}$. Given maps

$$c : \Omega \to \mathcal{C}, \quad \mathcal{A} : \mathcal{C} \to \mathrm{B}(X, Y^*), \quad \text{and } \mathcal{L} : \mathcal{C} \to Y^*,$$

let $\mathfrak{A} : L^2(\Omega; X) \to L^2(\Omega; Y)^*$ and $\mathfrak{L} \in L^2(\Omega; Y)^*$ be defined by

$$\big[\mathfrak{A}(u)\big](v) := \int_\Omega \big[\mathcal{A}_{c(\omega)} u(\omega)\big]\big(v(\omega)\big)\, \mathrm{d}\mathbb{P}(\omega) \quad \text{and} \quad \mathfrak{L}(v) := \int_\Omega \mathcal{L}_{c(\omega)}\big(v(\omega)\big)\, \mathrm{d}\mathbb{P}(\omega) \tag{2.1}$$

for $v \in L^2(\Omega; Y)$. Recall that a bounded linear map $X \to Y^*$ is equivalent to a sesquilinear (or bilinear) form on $X \times Y$; see e.g. [48, Lemma 2.1.38]. To keep notation compact, we write $\mathcal{A}_{c(\omega)} = (\mathcal{A} \circ c)(\omega)$ and $\mathcal{L}_{c(\omega)} = (\mathcal{L} \circ c)(\omega)$.

Remark 2.1 (Interpretation of the space $\mathcal{C}$). The space $\mathcal{C}$ is the 'space of inputs'. For the stochastic Helmholtz EDP in subsection 1.1 the space $\mathcal{C}$ is defined in Definition 4.5 below, but the upshot of this definition is that for any $\omega \in \Omega$ the triple $(A(\omega), n(\omega), f(\omega))$ is an element of $\mathcal{C}$. The maps c, $\mathcal{A}$, and $\mathcal{L}$ are given by $c = (A, n, f)$, $\mathcal{A} = a$, and $\mathcal{L} = L$, where a and L are given by (1.4) and (1.5) respectively and the equality $\mathcal{A} = a$ is meant in the sense of the one-to-one correspondence between $\mathrm{B}(X, Y^*)$ and sesquilinear forms on $X \times Y$.

The following three problems are the analogues in this general setting of Problems 1–3 in section 1.

Problem MAS (Measurable variational formulation almost surely). *Find a measurable function $u : \Omega \to X$ such that*

$$\mathcal{A}_{c(\omega)} u(\omega) = \mathcal{L}_{c(\omega)} \text{ in } Y^* \tag{2.2}$$

almost surely.

Problem SOAS (Second-order moment variational formulation almost surely). *Find $u \in L^2(\Omega; X)$ such that (2.2) holds almost surely.*

Problem SV (Stochastic variational formulation). *Find $u \in L^2(\Omega; X)$ such that*

$$\mathfrak{A} u = \mathfrak{L} \text{ in } L^2(\Omega; Y)^*. \tag{2.3}$$

Remark 2.2 (Immediate relationships between formulations). Since $L^2(\Omega; X) \subseteq \mathcal{B}(\Omega, X)$ (the space of all measurable functions $\Omega \to X$) it is immediate that if u solves Problem SOAS then every member of the equivalence class of u solves Problem MAS.

2.2. Conditions on $\mathcal{A}$, $\mathcal{L}$, and c. We now state the conditions under which we prove results about the equivalence of Problems MAS–SV.

Condition A1 ($\mathcal{A}$ is continuous). *The function $\mathcal{A} : \mathcal{C} \to \mathrm{B}(X, Y^*)$ is continuous, where we place the norm topology on X, the dual norm topology on Y^*, and the operator norm topology on $\mathrm{B}(X, Y^*)$.*

Condition A2 (Regularity of $\mathcal{A} \circ c$). *The map $\mathcal{A} \circ c \in L^\infty(\Omega; \mathrm{B}(X, Y^*))$.*

We note that Condition A2 is violated in the well-studied case of a log-normal coefficient κ for the stationary diffusion equation (1.2); in order to ensure the stochastic variational formulation is well-defined in this case, one must change the space of test functions as in [24, 41]

Condition L1 ($\mathcal{L}$ is continuous). *The function $\mathcal{L} : \mathcal{C} \to Y^*$ is continuous, where we place the dual norm topology on Y^*.*

Condition L2 (Regularity of $\mathcal{L} \circ c$). *The map $\mathcal{L} \circ c \in L^2(\Omega; Y^*)$.*

Condition C1 (c is measurable). *The function $c : \Omega \to \mathcal{C}$ is measurable.*

To state the next condition, we need to recall the following definition.

Definition 2.3 ($\mathbb{P}$-essentially separably valued [47, p26]). *Let $(S, \mathcal{T}_S)$ be a topological space. A function $h : \Omega \to S$ is* $\mathbb{P}$-essentially separably valued *if there exists $E \in \mathcal{F}$ such that $\mathbb{P}(E) = 1$ and $h(E)$ is contained in a separable subset of S.*

Condition C2 (c is $\mathbb{P}$-essentially separably valued). *The map $c : \Omega \to \mathcal{C}$ is $\mathbb{P}$-essentially separably valued.*

Remark 2.4 (Why do we need Condition C2?). The theory of Bochner spaces requires strong measurability of functions (see Definitions B.9 and B.14 below). However, the proof techniques used in this paper rely heavily on the measurability of functions (see Definition B.1 below). In separable spaces these two notions are equivalent (see Corollary B.19). However, some of the spaces we encounter (such as $L^\infty(D_R; \mathbb{R})$) are not separable. Therefore, in our arguments we use Condition C2 along with the Pettis Measurability Theorem (Theorem B.18 below) to conclude that measurable functions are strongly measurable.

Condition B (A priori bound almost surely). *There exist $C_j, f_j : \Omega \to \mathbb{R}$, $j = 1, \ldots, m$ such that $C_j f_j \in L^1(\Omega)$ for all $j = 1, \ldots, m$ and the bound*

$$\|u(\omega)\|_X^2 \leq \sum_{j=1}^{m} C_j(\omega) f_j(\omega) \tag{2.4}$$

holds almost surely.

Remark 2.5 (Notation in the a priori bound). We use the notation f_j in the right-hand side of (2.4) to emphasise the fact that typically these terms relate to the right-hand sides of the PDE in question. For the stochastic Helmholtz EDP, $m = 1$, $f_1 = \|f\|^2_{L^2(D)}$, and C_1 is given by (1.9).

Condition U (Uniqueness almost surely). $\ker\big(\mathcal{A}_{c(\omega)}\big) = \{0\}$ $\mathbb{P}$-*almost surely.*

The condition $\ker\big(\mathcal{A}_{c(\omega)}\big) = \{0\}$ $\mathbb{P}$-almost surely can be stated as: given $\mathcal{G} \in L^2(\Omega; Y)^*$, for $\mathbb{P}$-almost every $\omega \in \Omega$ the deterministic problem $\mathcal{A}_{c(\omega)} u_0 = \mathcal{G}$ has a unique solution,

2.3. Results on the equivalence of Problems MAS, SOAS, and SV.

Theorem 2.6 (Measurable solution implies second-order solution). *Under Condition B, if u solves Problem MAS then u solves Problem SOAS and satisfies the stochastic a priori bound*

$$\|u\|^2_{L^2(\Omega;X)} \le \sum_{j=1}^{m} \|C_j f_j\|_{L^1(\Omega)}. \tag{2.5}$$

Note that the right-hand side of the stochastic a priori bound (2.5) is the expectation of the right-hand side of the bound (2.4).

Lemma 2.7 (Stochastic variational formulation well-defined). *Under Conditions A1, A2, L1, L2, C1, and C2, the maps $\mathfrak{A}$ and $\mathfrak{L}$ defined by (2.1) are well-defined in the sense that*

$$[\mathfrak{A}(v_1)](v_2),\ \mathfrak{L}(v_2) < \infty \quad \textit{for all } v_1 \in L^2(\Omega; X),\ \textit{for all } v_2 \in L^2(\Omega; Y). \tag{2.6}$$

Theorem 2.8 (Second-order solution implies stochastic variational solution). *Under Conditions L1, L2, C1, and C2, if u solves Problem SOAS then u solves Problem SV.*

Theorem 2.9 (Stochastic variational solution implies second-order solution). *If Problem SV is well-defined and u solves Problem SV, then u solves Problem SOAS.*

Theorems 2.6, 2.8, and 2.9 and Lemma 2.7 are summarised in Figure 2.1.

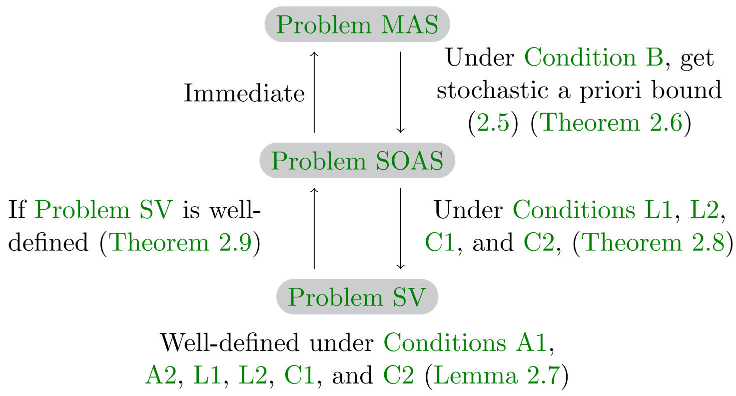

Figure 2.1. *The relationship between the variational formulations. An arrow from Problem P to Problem Q with Conditions R indicates 'under Conditions R, the solution of Problem P is a solution of Problem Q'*

Remark 2.10 (Condition L2 in Theorem 2.8). In Theorem 2.8 we could replace Condition L2 with Condition A2, and the result would still hold—see the proof for further details. However, Condition L2 is less restrictive than Condition A2, as it only requires L^2 integrability of $\mathcal{L} \circ c$ as opposed to essential boundedness of $\mathcal{A} \circ c$.

Lemma 2.11 (Showing uniqueness of the solution to Problems MAS–SV). *If Condition U holds, then*

1. *the solution to Problem MAS (if it exists) is unique up to modification on a set of $\mathbb{P}$-measure 0 in Ω,*
2. *the solution to Problem SOAS (if it exists) is unique in $L^2(\Omega; X)$, and*
3. *if Problem SV is well-defined, the solution to Problem SV (if it exists) is unique in $L^2(\Omega; X)$.*

Remark 2.12 (Informal discussion on the ideas behind the equivalence results). The diagram in Figure 2.1 summarises the relationships between the variational formulations, and the conditions under which they hold. Moving 'up' the left-hand side of the diagram, we prove a solution of Problem SV is a solution of Problem SOAS in Theorem 2.9; the key idea in this theorem is to use a particular set of test functions and the general measure-theory result of Lemma B.22 below; this approach was used for the stationary diffusion equation (1.2) with log-normal coefficients in [24], and for a wider class of coefficients in [41].

Moving 'down' the right-hand side, we prove a solution of Problem MAS is a solution of Problem SOAS in Theorem 2.6; the key part of this proof is that the bound in Condition B gives information on the integrability of the solution u. (In the case of (1.2) with uniformly coercive and bounded coefficient κ, the analogous integrability result follows from the Lax–Milgram theorem; [14, Proposition 2.4] proves an equivalent result for (1.2) with lognormal coefficient κ with an isotropic Lipschitz covariance function.) Proving a solution of Problem SOAS is a solution of Problem SV in Theorem 2.8 essentially amounts to posing conditions such that the quantities $\big[\mathcal{A}_{c(\omega)}(u(\omega))\big](v(\omega))$ and $\mathcal{L}_{c(\omega)}(v(\omega))$ are Bochner integrable for any $v \in L^2(\Omega; Y)$, so that (2.3) makes sense. Lemma 2.7 shows that the stronger property (2.6) holds, and requires stronger assumptions than Theorem 2.8, since the proof of Theorem 2.8 uses the additional information that u solves Problem SOAS.

Remark 2.13 (Changing the condition $u \in L^2(\Omega; X)$). Here we seek the solution $u \in L^2(\Omega; X)$ but we could instead require $u \in L^p(\Omega; X)$, for some $p > 0$ and require $\mathfrak{A}u = \mathfrak{L}$ in $L^q(\Omega; Y)^*$, for some $q > 0$ (i.e. use test functions in $L^q(\Omega; Y)$). In this case, the proof of Theorem 2.9 would be nearly identical, as the space $\mathcal{D}$ of test functions used there is a subset of $L^q(\Omega; Y)$ for all $q > 0$. One could also develop analogues of Theorems 2.6 and 2.8 and Lemma 2.7 in this setting—see e.g. [24, Theorem 3.20] for an example of this approach for the stationary diffusion equation with lognormal diffusion coefficient.

Remark 2.14 (Non-reliance on the Lax-Milgram theorem). The above results hold for an arbitrary sesquilinear form and hence are applicable to a wide variety of PDEs; their main advantage is that they apply to PDEs whose stochastic variational formulations are not coercive.

Remark 2.15 (Overview of how these results are applied to the Helmholtz equation in section 4). We obtain the results for the Helmholtz equation via the following steps (which could also be applied to other SPDEs fitting into this framework):

1. Define the map c (via $A, n,$ and f) such that for almost every $\omega \in \Omega$ there exists a solution of the deterministic Helmholtz EDP corresponding to $c(\omega)$.
2. Define $u : \Omega \to X$ to map ω to the solution of the deterministic problem corresponding

to $c(\omega)$.

3. Prove that Conditions A1, A2, L1, L2, C1, C2, B, and U hold, so that one can apply Theorems 2.6, 2.8, and 2.9 along with Lemmas 2.7 and 2.11 to show Problem 3 is well-defined and u is unique and satisfies Problems 1–3.

Steps 1 and 2 can be thought of as constructing a solution pathwise.

3. Proof of the results in section 2.

3.1. Preliminary lemmas. To simplify notation, we introduce the following definition.

Definition 3.1 (Pairing map). *For fixed $c : \Omega \to \mathcal{C}$, $\mathcal{A} : \Omega \to \mathrm{B}(X, Y^*)$, given $v : \Omega \to X$ we define the map $\pi_v : \Omega \to Y^*$ by*

$$\pi_v(\omega) := [(\mathcal{A} \circ c)(\omega)](v(\omega)). \tag{3.1}$$

A key ingredient in proving that the stochastic variational formulation is well-defined (Lemma 2.7) is showing that the maps π_u and $\mathcal{L} \circ c$ are measurable. Showing that $\mathcal{L} \circ c$ is measurable is straightforward (see Lemma 3.2 below), but showing that π_u is measurable is not. This is because $\mathcal{L} \circ c$ depends on ω only through its dependence on c, but π_u depends on ω through both the dependence of $\mathcal{A} \circ c$ on ω and the dependence of u on ω; it is this dual dependence that causes the extra complication.

Lemma 3.2 ($\mathcal{L} \circ c$ is measurable). *Under Conditions L1 and C1 the function $\mathcal{L} \circ c$ is measurable.*

Proof of Lemma 3.2. The map c is measurable (by Condition C1) and $\mathcal{L}$ is continuous (by Condition L1), therefore Lemma B.4 implies that $\mathcal{L} \circ c$ is measurable. ∎

Definition 3.3 (Product map). *For $v : \Omega \to X$, let $P_v : \Omega \to \mathrm{B}(X, Y^*) \times X$ be defined by $P_v(\omega) = \big((\mathcal{A} \circ c)(\omega), v(\omega)\big)$.*

Lemma 3.4 (Product map is measurable). *When $\mathrm{B}(X, Y^*) \times X$ is equipped with the product topology, if Conditions A1 and C1 hold, and if $v : \Omega \to X$ is measurable, then $P_v : \Omega \to \mathrm{B}(X, Y^*) \times X$ is measurable.*

Proof of Lemma 3.4. By the result on the measurability of the Cartesian product of measureable functions (Lemma B.6), P_v is measurable with respect to $\big(\mathcal{F}, \mathcal{B}\big(\mathrm{B}(X, Y^*)\big) \otimes \mathcal{B}(X)\big)$ (where $\mathcal{B}$ denotes the Borel σ-algebra—see Definition B.2), as both of the coordinate functions $\mathcal{A} \circ c$ and v are measurable. Since $\mathrm{B}(X, Y^*)$ and X are both metric spaces, they are both Hausdorff. As X is separable, Lemma B.7 on the product of Borel σ-algebras implies $\mathcal{B}\big(\mathrm{B}(X, Y^*)\big) \otimes \mathcal{B}(X) = \mathcal{B}\big(\mathrm{B}(X, Y^*) \times X\big)$. Hence P_v is measurable with respect to $\big(\mathcal{F}, \mathcal{B}\big(\mathrm{B}(X, Y^*) \times X\big)\big)$. ∎

Definition 3.5 (Evaluation map). *Let Z be a separable Banach space. The function $\eta_{Z^*} : B(X, Z^*) \times X \to Z^*$ is defined by*

$$\eta_{Z^*}\big((\mathcal{H}, v)\big) := \mathcal{H}(v) \quad \textit{for } \mathcal{H} \in B(X, Z^*) \textit{ and } v \in X. \tag{3.2}$$

Observe that the pairing, product, and evaluation maps (π_v, P_v, and, η_{Y^*} respectively) are related by $\pi_v = \eta_{Y^*} \circ P_v$.

Lemma 3.6 (Evaluation map is continuous). *Let Z be a separable Banach space. The map η_{Z^*} is continuous with respect to the product topology on $B(X, Z^*) \times X$ and the dual norm topology on Z^*.*

The proof of Lemma 3.6 is straightforward and omitted.

Lemma 3.7 (π_v is measurable). *If Conditions A1 and C1 hold and v is measurable, then the function π_v as defined by (3.1) is measurable.*

Proof of Lemma 3.7. By Lemma 3.4 P_v is measurable and by Lemma 3.6 η_{Y^*} is continuous. Therefore Lemma B.4 implies that $\pi_v = \eta_{Y^*} \circ P_v$ is measurable. ∎

3.2. Proofs of Theorems 2.6, 2.8, and 2.9 and Lemmas 2.7 and 2.11.

Proof of Theorem 2.6. We need to show $u : \Omega \to X$ is strongly measurable, satisfies the bound (2.5), and therefore is Bochner integrable and is in the space $L^2(\Omega; X)$. Our plan is to use Corollary B.12 to show u is Bochner integrable, and establish (2.5) as a by-product. Since u solves Problem MAS, u is measurable. As X is separable, it follows from Corollary B.19 that u is strongly measurable. Define $N : X \to \mathbb{R}$ by $N(v) := \|v\|_X^2$. Since N is continuous, Lemma B.4 implies $N \circ u : \Omega \to \mathbb{R}$ is measurable. Therefore, since both the left- and right-hand sides of (2.4) are measurable and (2.4) holds for almost every $\omega \in \Omega$ we can integrate (2.4) over Ω with respect to $\mathbb{P}$ and obtain

$$\int_\Omega \|u(\omega)\|_X^2 \, \mathrm{d}\mathbb{P}(\omega) \le \sum_{j=1}^m \|C_j f_j\|_{L^1(\Omega)}, \tag{3.3}$$

the right-hand side of which is finite since Condition B includes that $C_j f_j \in L^1(\Omega)$ for all $j = 1, \ldots, m$. Since u is strongly measurable, the bound (3.3) and Corollary B.12 with $p = 2$ imply that u is Bochner integrable. The norm $\|u\|_{L^2(\Omega;X)}$ is thus well-defined by Definition B.13 and (3.3) shows that (2.5) holds, and so in particular $\|u\|_{L^2(\Omega;X)} < \infty$. ∎

Proof of Lemma 2.7. We must show that for any $v_1 \in L^2(\Omega; X)$ and any $v_2 \in L^2(\Omega; Y)$:

- The quantities $\left[\mathcal{A}_{c(\omega)} v_1(\omega)\right]\big(v_2(\omega)\big)$ and $\mathcal{L}_{c(\omega)}\big(v_2(\omega)\big)$ are Bochner integrable, so that the definitions of $\mathfrak{A}$ and $\mathfrak{L}$ as integrals over Ω make sense.
- The maps $\mathfrak{A}(v_1)$ and $\mathfrak{L}$ are linear and bounded on $L^2(\Omega; Y)$, that is, $\mathfrak{A} : L^2(\Omega; X) \to L^2(\Omega; Y)^*$ and $\mathfrak{L} \in L^2(\Omega; Y)^*$.

It follows from these two points that $\mathfrak{A}$ and $\mathfrak{L}$ are well-defined. Thanks to the groundwork laid in subsection 3.1, the measurability of $\left[\mathcal{A}_{c(\omega)} v_1(\omega)\right]\big(v_2(\omega)\big)$ and $\mathcal{L}_{c(\omega)}\big(v_2(\omega)\big)$ follows from Lemmas 3.2 and 3.7 (which need Conditions A1–C2). Their $\mathbb{P}$-essential separability follows from Conditions A1–C2 and Lemma B.20 and thus their strong measurability follows from Corollary B.19 on the equivalence of measurability and strong measurability when the image is separable. Their Bochner integrability then follows from the Bochner integrability condition in Theorem B.11 (with $V = \mathbb{F}$) and the Cauchy–Schwartz inequality since

$$\int_\Omega \left|\mathcal{L}_{c(\omega)}\big(v_2(\omega)\big)\right| \mathrm{d}\mathbb{P}(\omega) \le \|\mathcal{L} \circ c\|_{L^2(\Omega;Y^*)} \|v_2\|_{L^2(\Omega;Y)}, \tag{3.4}$$

which is finite by Condition L2, and

$$\int_\Omega \left|\left[\mathcal{A}_{c(\omega)} v_1(\omega)\right]\big(v_2(\omega)\big)\right| \mathrm{d}\mathbb{P}(\omega) \le \|\mathcal{A} \circ c\|_{L^\infty(\Omega;\mathrm{B}(X,Y^*))} \|v_1\|_{L^2(\Omega;X)} \|v_2\|_{L^2(\Omega;Y)}, \tag{3.5}$$

which is finite by Condition A2. We now show $\mathfrak{L} \in L^2(\Omega; Y)^*$ and $\mathfrak{A} : L^2(\Omega; X) \to L^2(\Omega; Y)^*$. Observe that $|\mathfrak{L}(v_2)| \leq \int_\Omega \left|\mathcal{L}_{c(\omega)}(v_2(\omega))\right| \mathrm{d}\mathbb{P}(\omega)$ and $|[\mathfrak{A}(v_1)](v_2)| \leq \int_\Omega \left|\left[\mathcal{A}_{c(\omega)} v_1(\omega)\right](v_2(\omega))\right| \mathrm{d}\mathbb{P}(\omega)$ and thus by (3.4) and (3.5) $\mathfrak{L}$ and $\mathfrak{A}(v_1)$ are bounded. They are clearly linear, and so it follows that $\mathfrak{L} \in L^2(\Omega; Y)^*$ and $\mathfrak{A}(v_1) \in L^2(\Omega; Y)^*$, i.e., $\mathfrak{A} : L^2(\Omega; X) \to L^2(\Omega; Y)^*$. ∎

Proof of Theorem 2.8. In order to show that u solves Problem SV, we must show:

1. either the functional $\mathfrak{L} \in L^2(\Omega; Y)^*$ or the functional $\mathfrak{A}(u) \in L^2(\Omega; Y)^*$, and
2. the equality (2.3) holds.

For Point 1 we show that $\mathfrak{L} \in L^2(\Omega; Y)^*$, (since this is easier than showing $\mathfrak{A}(u) \in L^2(\Omega; Y)^*$); in fact the proof of this is contained in the proof of Lemma 2.7.

For Point 2, since u solves Problem SOAS, for $\mathbb{P}$-almost every $\omega \in \Omega$ we have $\mathcal{A}_{c(\omega)} u(\omega) = \mathcal{L}_{c(\omega)}$ in Y^*. Hence, for any $v \in L^2(\Omega; Y)$ we have

$$\left[\mathcal{A}_{c(\omega)} u(\omega)\right]\big(v(\omega)\big) = \mathcal{L}_{c(\omega)}\big(v(\omega)\big) \tag{3.6}$$

for $\mathbb{P}$-almost every $\omega \in \Omega$. Since $\mathfrak{L} \in L^2(\Omega; Y)^*$, the right-hand side of (3.6) is a strongly measurable function with finite integral. Hence the left-hand side of (3.6) is as well, and we integrate over Ω to conclude $\left[\mathfrak{A}u\right](v) = \mathfrak{L}(v)$ for all $v \in L^2(\Omega; Y)$, i.e., $\mathfrak{A}u = \mathfrak{L}$ in $L^2(\Omega; Y)^*$. ∎

The following lemma is needed for the proof of Theorem 2.9.

Lemma 3.8. *Let $\delta : \Omega \times Y \to \mathbb{F}$. For $y \in Y$, define $\Omega_y := \{\omega \in \Omega : \delta(\omega, y) = 0\}$ and define $\widetilde{\Omega} := \{\omega \in \Omega : \delta(\omega, y) = 0$ for all $y \in Y\}$. If*

- *for all $\omega \in \Omega$, $\delta(\omega, \cdot)$ is a continuous functional on Y and*
- *for all $y \in Y$, the map $\delta(\cdot, y) : \Omega \to \mathbb{F}$ is measurable and $\mathbb{P}(\Omega_y) = 1$,*

then $\mathbb{P}(\widetilde{\Omega}) = 1$.

Proof of Lemma 3.8. We must show that the set $\widetilde{\Omega} \in \mathcal{F}$, and $\mathbb{P}(\widetilde{\Omega}) = 1$. Observe that, for any $y \in Y$, the set $\Omega_y \in \mathcal{F}$, since $\Omega_y = \delta(\cdot, y)^{-1}(\{0\})$, which is the preimage under a measurable map of a measurable set.

Since Y is a Hilbert space, it is separable, and therefore it has a countable dense subset $(y_n)_{n\in\mathbb{N}}$. We will show that $\mathbb{P}(\cap_{n\in\mathbb{N}} \Omega_{y_n}) = 1$ and $\widetilde{\Omega} = \cap_{n\in\mathbb{N}} \Omega_{y_n}$. The set $\cap_{n\in\mathbb{N}} \Omega_{y_n} \in \mathcal{F}$, as $\mathcal{F}$ is a σ-algebra and $\mathbb{P}\big(\cup_{n\in\mathbb{N}} \Omega_{y_n}^c\big) \leq \sum_{n\in\mathbb{N}} \mathbb{P}\big(\Omega_{y_n}^c\big) = 0$, and hence $\mathbb{P}(\cap_{n\in\mathbb{N}} \Omega_{y_n}) = 1$. To next show $\widetilde{\Omega} = \cap_{n\in\mathbb{N}} \Omega_{y_n}$ we observe that $\widetilde{\Omega} = \cap_{y\in Y} \Omega_y$ and $\cap_{y\in Y} \Omega_y \subseteq \cap_{n\in\mathbb{N}} \Omega_{y_n}$. It therefore suffices to show $\cap_{n\in\mathbb{N}} \Omega_{y_n} \subseteq \cap_{y\in Y} \Omega_y$ to conclude $\widetilde{\Omega} = \cap_{n\in\mathbb{N}} \Omega_{y_n}$.

Fix $y \in Y$. By density of $(y_n)_{n\in\mathbb{N}}$, there exists a subsequence $(y_{n_m})_{m\in\mathbb{N}}$ such that $y_{n_m} \to y$ as $m \to \infty$. Fix $\omega \in \cap_{n\in\mathbb{N}} \Omega_{y_n}$. Note that $\omega \in \cap_{m\in\mathbb{N}} \Omega_{y_{n_m}}$; that is, for all $m \in \mathbb{N}$, $\delta(\omega, y_{n_m}) = 0$. As $\delta(\omega, \cdot)$ is a continuous function on Y, $\delta(\omega, y_{n_m}) \to \delta(\omega, y)$ as $m \to \infty$. But as previously noted, $\delta(\omega, y_{n_m}) = 0$ for all $m \in \mathbb{N}$. Hence we must have $\delta(\omega, y) = 0$, and thus $\omega \in \Omega_y$. Since $\omega \in \cap_{n\in\mathbb{N}} \Omega_{y_n}$ was arbitrary, it follows that $\cap_{n\in\mathbb{N}} \Omega_{y_n} \subseteq \Omega_y$, and since $y \in Y$ was arbitrary, it follows that $\cap_{n\in\mathbb{N}} \Omega_{y_n} \subseteq \cap_{y\in Y} \Omega_y$ as required. ∎

Proof of Theorem 2.9. Let $u \in L^2(\Omega; X)$ solve Problem SV. We need to show that u solves Problem SOAS. Observe that u solving Problem SOAS means $\mathcal{A}_{c(\omega)}(u(\omega)) = \big(\mathcal{L}_{c(\omega)}\big)(\omega)$ in Y^* for almost every $\omega \in \Omega$. We now use an idea from [24, Theorem 3.3]. Our plan is to use test functions of the form $y\mathbb{1}_E$, where $y \in Y$ and $E \in \mathcal{F}$ to reduce Problem SV to the statement

$$\int_E \left[\mathcal{A}_{c(\omega)}\big(u(\omega)\big)\right]\big(y(\omega)\big)\,\mathrm{d}\mathbb{P}(\omega) = \int_E \left[\big(\mathcal{L}_{c(\omega)}\big)(\omega)\right]\big(y(\omega)\big)\,\mathrm{d}\mathbb{P}(\omega) \quad \text{for all } E \in \mathcal{F}$$

and then show this implies u satisfies Problem SOAS via Lemma B.22.

First let $\mathcal{D} := \{y\mathbb{1}_E : y \in Y, E \in \mathcal{F}\}$ and observe that the elements of $\mathcal{D}$ are maps from Ω to Y. The fact that $\mathcal{D} \subseteq L^2(\Omega; Y)$ follows via the following three steps:

1. The elements of $\mathcal{D}$ are measurable, indeed the indicator function of a measurable set is a measurable function $\Omega \to \mathbb{R}$, and multiplication by $y \in Y$ is a continuous function $\mathbb{R} \to Y$. Hence elements of $\mathcal{D}$ are measurable by Lemma B.4.
2. As Y is a separable Hilbert space, it follows from Corollary B.19 that the elements of $\mathcal{D}$ are strongly measurable.
3. $\|y\mathbb{1}_E\|_{L^2(\Omega;Y)} = \sqrt{\mathbb{P}(E)}\|y\|_Y < \infty$ for all $y \in Y, E \in \mathcal{F}$.

Since Problem SV is well-defined, and u solves Problem SV, and $\mathcal{D} \subseteq L^2(\Omega; Y)$, we have that $[\mathfrak{A}u](v) = \mathfrak{L}(v)$ for all $v \in \mathcal{D}$. Therefore, we have

$$\int_\Omega \left[\mathcal{A}_{c(\omega)}(u(\omega))\right](y\mathbb{1}_E(\omega))\,\mathrm{d}\mathbb{P}(\omega) = \int_\Omega \left[\mathcal{L}_{c(\omega)}\right](y\mathbb{1}_E(\omega))\,\mathrm{d}\mathbb{P}(\omega) \tag{3.7}$$

for all $y \in Y$ and $E \in \mathcal{F}$. If we define $\delta : \Omega \times Y \to \mathbb{F}$ by $\delta(\omega, y) := \left[\mathcal{A}_{c(\omega)}(u(\omega)) - \mathcal{L}_{c(\omega)}\right](y)$ then, by the definition of $\mathbb{1}_E$, (3.7) becomes

$$\int_E \delta(\omega, y)\,\mathrm{d}\mathbb{P}(\omega) = 0 \quad \text{for all } E \in \mathcal{F}. \tag{3.8}$$

To conclude u solves Problem SOAS we must show $\delta(\omega, y) = 0$ for all $y \in Y$, almost surely. We will use Lemma B.22, so the first step is to show that for all $y \in Y$ $\delta(\cdot, y)$ is Bochner integrable. This follows from the fact that Problem SV is well-defined, and thus the quantities $\left[\mathcal{A}_{c(\omega)}v_1(\omega)\right]\big(v_2(\omega)\big)$ and $\mathcal{L}_{c(\omega)}\big(v_2(\omega)\big)$ are Bochner integrable for any $v_1 \in L^2(\Omega; X), v_2 \in L^2(\Omega; Y)$. In particular, they are Bochner integrable when $v_1 = u$, and $v_2 = y\mathbb{1}_E$ and thus their difference δ is Bochner integrable. Secondly, $\delta(\omega, \cdot)$ is a continuous function on Y since $\mathcal{A}_{c(\omega)}(u(\omega))$ and $\big(\mathcal{L}_{c(\omega)}\big)(\omega) \in Y^*$, for all $\omega \in \Omega$.

We now show $\delta(\omega, y) = 0$ for all $y \in Y$, almost surely. For $y \in Y$ define the set $\Omega_y := \{\omega \in \Omega : \delta(\omega, y) = 0\}$; by (3.8) and Lemma B.22 we have that $\mathbb{P}(\Omega_y) = 1$ for all $y \in Y$. By Lemma 3.8, $\delta(\omega, y) = 0$ for all $y \in Y$, almost surely, that is, $\mathcal{A}_{c(\omega)}u(\omega) = \mathcal{L}_{c(\omega)}$ almost surely; it follows that u solves Problem SOAS. ■

Remark 3.9 (Connection with the argument in [41, Remark 2.2]). The argument in Lemma 3.8 and the final part of Theorem 2.9 closely mirrors the result in [41, Remark 2.2]. Indeed, we prove in general that $\mathbb{P}\big(\delta(\omega, y) = 0\big) = 1$ for all $y \in Y$ implies $\mathbb{P}\big(\delta(\omega, y) = 1$ for all $y \in Y\big) = 1$, and [41, Remark 2.2] shows an analogous result for the stationary diffusion equation (1.2) with non-uniformly coercive and unbounded coefficient κ.

Proof of Lemma 2.11. *Proof of Part* 1. Suppose $u_1, u_2 : \Omega \to X$ solve Problem MAS. Let $E = \{\omega \in \Omega : u_1(\omega) \neq u_2(\omega)\}$. Denote by E_1 and E_2 the sets (of measure zero) where the variational problems for u_1 and u_2 fail to hold, i.e. $E_1, E_2 \in \mathcal{F}$ with $\mathbb{P}(E_1) = \mathbb{P}(E_2) = 0$ and

$\mathcal{A}_{c(\omega)}(u_1(\omega)) \neq \mathcal{L}_{c(\omega)}$ iff $\omega \in E_1$, and $\mathcal{A}_{c(\omega)}(u_2(\omega)) \neq \mathcal{L}_{c(\omega)}$ iff $\omega \in E_2$. As $\ker(\mathcal{A}_{c(\omega)}) = \{0\}$ $\mathbb{P}$-almost surely, there exists $E_3 \in \mathcal{F}$ such that $\mathbb{P}(E_3) = 0$ and $\ker(\mathcal{A}_{c(\omega)}) \neq \{0\}$ iff $\omega \in E_3$. We claim $E \subseteq E_1 \cup E_2 \cup E_3$. Indeed, if $u_1(\omega) \neq u_2(\omega)$ then either: (i) at least one of u_1 and u_2 does not solve Problem MAS at ω or (ii) u_1 and u_2 both solve Problem MAS at ω, but $\ker(\mathcal{A}_{c(\omega)}) \neq \{0\}$. Since $\mathbb{P}(E_j) = 0, j = 1, 2, 3$, we have $\mathbb{P}(E_1 \cup E_2 \cup E_3) = 0$. Therefore $E \in \mathcal{F}$ and $\mathbb{P}(E) = 0$ since $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space; hence $u_1 = u_2$ almost surely.

Proof of Part 2. By Remark 2.2, if $u_1, u_2 \in L^2(\Omega; X)$ solve Problem SOAS, then all the representatives of the equivalence classes of u_1 and u_2 solve Problem MAS. Hence, by Part 1, any representatives of u_1 and u_2 differ only on some set (depending on the representatives) of $\mathbb{P}$-measure zero in Ω. Therefore $u_1 = u_2$ in $L^2(\Omega; X)$, by definition of $L^2(\Omega; X)$.

Proof of Part 3. As Problem SV is well-defined, by Remark 2.2 and Theorem 2.9, if u_1 and u_2 solve Problem SV, then u_1 and u_2 also solve Problem MAS. We then repeat the reasoning in the proof of Part 2 to show $u_1 = u_2$ in $L^2(\Omega; X)$. ∎

4. Proofs of Theorems 1.4 and 1.8.

In subsection 4.1 we place the Helmholtz stochastic EDP into the framework developed in section 2. In subsection 4.2 we give sufficient conditions for the Helmholtz stochastic EDP to satisfy Conditions A1, L1, and C1, etc.. In subsection 4.3 we apply the general theory developed in section 2 to prove Theorems 1.4 and 1.8.

4.1. Placing the Helmholtz stochastic EDP into the framework of section 2.

Recall $R > 0$ is fixed. We let $X = Y = H^1_{0,D}(D_R)$ and define the norm $\|v\|^2_{1,k} := \|\nabla v\|^2_{L^2(D_R)} + k^2\|v\|^2_{L^2(D_R)}$ on $H^1_{0,D}(D_R)$. Throughout this section, A_0, n_0, and f_0 will be deterministic functions. Recall that since the supports of $1 - n$, $I - A$, and f are compactly contained in B_R, we can consider A, n, and f as functions on D_R rather than on D_+. In order to define the space $\mathcal{C}$ and the maps $c, \mathcal{A}$, and $\mathcal{L}$ we define the following function spaces on D_R.

Definition 4.1 (Compact-support spaces). *Let*

$$
\begin{aligned}
L^2_R(D_R) &:= \left\{f_0 \in L^2(D_R) : \operatorname{ess\,supp}(f_0) \subset\subset B_R\right\},\\
L^\infty_{R,\min}(D_R;\mathbb{R}) &:= \big\{n_0 \in L^\infty(D_R;\mathbb{R}) : \operatorname{ess\,supp}(1-n_0) \subset\subset B_R,\\
&\qquad \textit{there exists } \alpha_{n_0} > 0 \textit{ such that } n_0(\mathbf{x}) \geq \alpha_{n_0} \textit{ almost everywhere } \big\},\\
L^\infty_{R,\min}\left(D_R;\mathbb{R}^{d\times d}\right) &:= \Big\{A_0 \in L^\infty\left(D_R;\mathbb{R}^{d\times d}\right) : A_0(\mathbf{x}) \textit{ is symmetric almost everywhere,}\\
&\qquad \operatorname{ess\,supp}(I - A_0) \subset\subset B_R, \textit{there exists } \alpha_{A_0} > 0 \textit{ s. t. } \alpha_{A_0} \leq A_0(\mathbf{x})\\
&\qquad \textit{almost everywhere, in the sense of quadratic forms}\Big\}, \textit{ and}\\
W^{1,\infty}_{R,\min}\left(D_R;\mathbb{R}^{d\times d}\right) &:= \left\{A_0 \in L^\infty_{R,\min}\left(D_R;\mathbb{R}^{d\times d}\right) : A_0 \in W^{1,\infty}\left(D_R;\mathbb{R}^{d\times d}\right)\right\}.
\end{aligned}
$$

Observe that the norm on $L^\infty(D_R;\mathbb{R})$ induces a metric on $L^\infty_{R,\min}(D_R;\mathbb{R})$, and similarly for $L^\infty_R\left(D_R;\mathbb{R}^{d\times d}\right)$, $W^{1,\infty}_{R,\min}\left(D_R;\mathbb{R}^{d\times d}\right)$, and $L^2_R(D_R)$. These spaces are not vector spaces, and are not complete, but completeness and being a vector space is not required in what follows—we only need them to be metric spaces.

Definition 4.2 (Deterministic form and functional).
For $(A_0, n_0, f_0) \in L^\infty_R\left(D_R;\mathbb{R}^{d\times d}\right) \times L^\infty_{R,\min}(D_R;\mathbb{R}) \times L^2_R(D_R)$ let the sesquilinear form a_{A_0,n_0}

on $H^1_{0,D}(D_R) \times H^1_{0,D}(D_R)$ *and the antilinear functional* L_{f_0} *on* $H^1_{0,D}(D_R)$ *be given by*

$$a_{A_0,n_0}(v_1,v_2) := \int_{D_R} \Big((A_0 \nabla v_1) \cdot \nabla \overline{v_2} \rangle - k^2 n_0 \, v_1 \, \overline{v_2} \Big) \, \mathrm{d}\lambda - \langle T_R \gamma v_1, \gamma v_2 \rangle_{\Gamma_R}, \quad \textit{and}$$

$$L_{f_0}(v_2) := \int_{D_R} f_0 \, \overline{v_2} \, \mathrm{d}\lambda, \qquad \textit{for } v_1, v_2 \in H^1_{0,D}(D_R).$$

Problem 4.3 (Helmholtz EDP). For $(A_0, n_0, f_0) \in L^\infty_R(D_R; \mathbb{R}^{d\times d}) \times L^\infty_R(D_R; \mathbb{R}) \times L^2_R(D_R)$ find $u_0 \in H^1_{0,D}(D_R)$ such that $a_{A_0,n_0}(u_0, v) = L_{f_0}(v)$ for all $v \in H^1_{0,D}(D_R)$.

Definition 4.4 (d_∞ metric). *Let* $(X_1, d_1), \ldots, (X_m, d_m)$ *be metric spaces. The* d_∞ metric *on the Cartesian product* $X_1 \times \cdots \times X_m$ *is defined by*

$$d_\infty((x_1, \ldots, x_m), (y_1, \ldots, y_m)) := \max_{j=1,\ldots,m} d_j(x_j, y_j).$$

Definition 4.5 (The input space $\mathcal{C}$). *We let* $\mathcal{C} := W^{1,\infty}_{R,\min}(D_R; \mathbb{R}^{d\times d}) \times L^\infty_{R,\min}(D_R; \mathbb{R}) \times L^2_R(D_R)$ *with topology given by the* d_∞ *metric.*

Definition 4.6 (The input map c). *Define* $c : \Omega \to \mathcal{C}$ *by* $c(\omega) = (A(\omega), n(\omega), f(\omega))$.

Definition 4.7 (The maps $\mathcal{A}$ and $\mathcal{L}$ for the Helmholtz stochastic EDP). *Let*

$$\mathcal{A}((A_0, n_0, f_0)) := a_{A_0,n_0} \quad \textit{and} \quad \mathcal{L}((A_0, n_0, f_0)) := L_{f_0}, \tag{4.1}$$

where the definition of $\mathcal{A}$ *is understood in terms of the equivalence between* $\mathrm{B}(X, Y^*)$ *and sesquilinear forms on* $X \times Y$.

4.2. Verifying the Helmholtz stochastic EDP satisfies the conditions in section 2.

Lemma 4.8 (Conditions C1 and C2 for Helmholtz stochastic EDP). *If* $A, n,$ *and* f *are strongly measurable, then* c *defined by Definition* 4.6 *satisfies Conditions* C1 *and* C2.

Proof. Since $A, n,$ and f are strongly measurable, by Theorem B.18 they are measurable and $\mathbb{P}$-essentially separably valued. By Lemma B.6, it follows that c is measurable, so c satisfies Condition C1. By Lemma B.23, it follows that c is $\mathbb{P}$-essentially separably valued, so c satisfies Condition C2. ■

Lemma 4.9 (Conditions A1 and L1 for Helmholtz stochastic EDP). *The maps* $\mathcal{A}$ *and* $\mathcal{L}$ *given by* (4.1) *satisfy Conditions* A1 *and* L1.

Proof of Lemma 4.9. We need to show that if $(A_m, n_m, f_m) \to (A_0, n_0, f_0)$ in $\mathcal{C}$ then $\mathcal{A}((A_m, n_m, f_m)) \to \mathcal{A}((A_0, n_0, f_0))$ in $\mathrm{B}(X, Y^*)$, and similarly for $\mathcal{L}$. By the Cauchy–Schwarz inequality we have, for $v_1 \in X, v_2 \in Y$,

$$\begin{aligned}
&\left| \Big[\big[\mathcal{A}(A_m, n_m, f_m) - \mathcal{A}(A_0, n_0, f_0) \big](v_1) \Big](v_2) \right| \\
&\qquad \le \|A_m - A_0\|_{L^\infty(D_R)} \|\nabla v_1\|_{L^2(D_R)} \|\nabla v_2\|_{L^2(D_R)} \\
&\qquad\qquad + k^2 \|n_m - n_0\|_{L^\infty(D_R;\mathbb{R})} \|v_1\|_{L^2(D_R)} \|v_2\|_{L^2(D_R)} \\
&\qquad \le 2 d_\infty((A_m, n_m, f_m), (A_0, n_0, f_0)) \|v_1\|_{1,k} \|v_2\|_{1,k},
\end{aligned}$$

Hence if $(A_m, n_m, f_m) \to (A_0, n_0, f_0)$ in $\mathcal{C}$, then $\mathcal{A}((A_m, n_m, f_m)) \to \mathcal{A}((A_0, n_0, f_0))$ in $\mathrm{B}(X, Y^*)$. We also have

$$\left|\left[\mathcal{L}((A_m, n_m, f_m),) - \mathcal{L}((A_0, n_0, f_0))\right](v_2)\right| = \left|\int_{D_R} (f_m - f_0)\overline{v_2}\, \mathrm{d}\lambda\right| \le \|f_m - f_0\|_{L^2(D_R)} \frac{\|v_2\|_{1,k}}{k}.$$

Hence if $(A_m, n_m, f_m) \to (A_0, n_0, f_0)$ in $\mathcal{C}$, then $\mathcal{L}((A_m, n_m, f_m)) \to \mathcal{L}((A_0, n_0, f_0))$ in Y^*. ■

Definition 4.10 (The solution operator $\mathcal{S}$). *Define $\mathcal{S} : \mathcal{C} \to H^1_{0,D}(D_R)$ by letting $\mathcal{S}(A_0, n_0, f_0) \in H^1_{0,D}(D_R)$ be the solution of the Helmholtz EDP (Problem 4.3).*

Theorem 4.11 ($\mathcal{S}$ is well defined). *For $(A_0, n_0, f_0) \in \mathcal{C}$ the solution $\mathcal{S}((A_0, n_0, f_0))$ of the Helmholtz EDP (Problem 4.3) exists, is unique, and depends continuously on f_0.*

Proof of Theorem 4.11. Since $\Re(-\langle T_R\gamma v, \gamma v\rangle_{\Gamma_R}) \ge 0$ for all $v \in H^1_{0,D}(D_R)$ (see, e.g. [42, Theorem 2.6.4]), a_{A_0,n_0} satisfies a Gårding inequality. Since the inclusion $H^1_{0,D}(D_R) \hookrightarrow L^2(D_R)$ is compact, Fredholm theory shows that uniqueness implies well-posedness (see, e.g. [39, Theorem 2.34]). Since A is Lipschitz and n is L^∞, uniqueness follows from the unique continuation results in [33, 23]; see [26, Section 2] for these results specifically applied to Helmholtz problems. ■

Lemma 4.12 (Continuity of solution operator for Helmholtz stochastic EDP). *For the Helmholtz stochastic EDP, the solution operator $\mathcal{S} : \mathcal{C} \to H^1_{0,D}(D_R)$ is continuous.*

Sketch Proof of Lemma 4.12. Let $(A_0, n_0, f_0), (A_1, n_1, f_1) \in \mathcal{C}$, with $\mathcal{S}((A_0, n_0, f_0)) = u_0$ and $\mathcal{S}((A_1, n_1, f_1)) = u_1$. Then for any $v \in H^1_{0,D}(D_R)$ we have, for $j = 0, 1$,

$[[\mathcal{A}((A_j, n_j, f_j))](u_j)](v) = [\mathcal{L}((A_j, n_j, f_j))](v)$.

Continuity of $\mathcal{S}$ then follows from:

1. Deriving the Helmholtz equation with coefficients A_0 and n_0 satisfied by $u_d := u_0 - u_1$.
2. Recalling that the well-posedness result of Theorem 4.11 holds when $f_0 \in L^2_{\mathbb{R}}(D_R)$ is replaced by a right-hand side in $(H^1_{0,D}(D_R))^*$; see, e.g., [39, Theorem 2.34].
3. Applying the result in Point 2 to obtain a bound $\|u_d\|_{1,k} \le C(A_0, n_0)\|F\|_{\left(H^1_{0,D}(D_R)\right)^*}$.
4. Showing $\|F\|_{\left(H^1_{0,D}(D_R)\right)^*}$ depends on $\|\nabla u_1\|_{L^2(D_R)}$, $\|u_1\|_{L^2(D_R)}$, $\|A_1 - A_0\|_{L^\infty\left(D_R;\mathbb{R}^{d\times d}\right)}$, $\|n_1 - n_0\|_{L^\infty(D_R;\mathbb{R})}$, and $\|f_0 - f_1\|_{L^2(D)}$.
5. Eliminating the dependence on u_1 by writing $u_1 = u_0 - u_d$ and moving terms in u_d to the left-hand side, to obtain a bound on u_d of the form
$$\begin{aligned}&\|\nabla u_d\|_{L^2(D_R)} + k\|u_d\|_{L^2(D_R)}\\ &\qquad \le \widetilde{C}\Big(u_0, A_0, n_0, \|A_1 - A_0\|_{L^\infty\left(D_R;\mathbb{R}^{d\times d}\right)}, \|n_1 - n_0\|_{L^\infty(D_R;\mathbb{R})}, \|f_0 - f_1\|_{L^2(D_R)}\Big).\end{aligned}$$
6. Concluding that $u_d \to 0$ in $H^1_{0,D}(D_R)$ as $(A_1, n_1, f_1) \to (A_0, n_0, f_0)$ in $\mathcal{C}$. ■

Lemma 4.13 (Condition U for the Helmholtz stochastic EDP). *The Helmholtz stochastic EDP satisfies Condition U.*

Proof of Lemma 4.13. This condition holds immediately from Theorem 4.11. ■

To prove that Condition B holds for the Helmholtz stochastic EDP, we first state the deterministic analogues of Condition 1.6 and Theorem 1.8.

Condition 4.14 (Nontrapping condition for Helmholtz EDP [25, Condition 2.4]). $d = 2, 3$, D_- is star-shaped with respect to the origin, $A_0 \in W^{1,\infty}(D_R; \mathbb{R}^{d\times d})$, $n_0 \in W^{1,\infty}(D_R; \mathbb{R})$, and there exist $\tau_1, \tau_2 > 0$ such that, for almost every $\mathbf{x} \in D_+$, $A_0(\mathbf{x}) - (\mathbf{x}\cdot\nabla)A_0(\mathbf{x}) \geq \tau_1$ and $n_0(\mathbf{x}) + \mathbf{x}\cdot\nabla n_0(\mathbf{x}) \geq \tau_2$, where the first inequality holds in the sense of quadratic forms.

Theorem 4.15 (Well-posedness of the Helmholtz EDP under Condition 4.14 [25, Theorem 2.5]). *Let $(A_0, n_0, f_0) \in \mathcal{C}$ and suppose A_0 and n_0 satisfy Condition 4.14. Then the solution of the Helmholtz EDP (Problem 4.3) exists and is unique. Furthermore, given $k_0 > 0$ for all $k \geq k_0$, the solution u_0 of the Helmholtz EDP satisfies the bound*

$$\tau_1\|\nabla u_0\|^2_{L^2(D_R)} + \tau_2 k^2\|u_0\|^2_{L^2(D_R)} \leq C_1\|f_0\|^2_{L^2(D_R)}, \text{ where } C_1 := 4\left[\frac{R^2}{\tau_1} + \frac{1}{\tau_2}\left(R + \frac{d-1}{2k_0}\right)^2\right]. \tag{4.2}$$

We can now prove Condition B holds for the Helmholtz stochastic EDP.

Lemma 4.16 (Condition B for Helmholtz stochastic EDP). *If Conditions 1.3 and 1.6 hold, then Condition B holds for the Helmholtz stochastic EDP.*

Proof of Lemma 4.16. As Condition 1.6 holds, Condition 4.14 holds for $\mathbb{P}$-almost every $\omega \in \Omega$ (with $A_0 = A(\omega)$, $n_0 = n(\omega)$, $\tau_1 = \mu_1(\omega)$, and $\tau_2 = \mu_2(\omega)$). Hence, by Theorem 4.15 the bound (2.4) holds for all $k \geq k_0$, with $X = H^1_{0,D}(D_R), m = 1$,

$$C_1(\omega) = \frac{4}{\min\{\mu_1(\omega), \mu_2(\omega)\}}\left[\frac{R^2}{\mu_1(\omega)} + \frac{1}{\mu_2(\omega)}\left(R + \frac{d-1}{2k_0}\right)^2\right],$$

and $f_1 = \|f(\omega)\|^2_{L^2(D_R)}$. It now remains to show that $C_1\,\|f\|^2_{L^2(D_R)} \in L^1(\Omega)$. We first show $C_1\,\|f\|^2_{L^2(D_R)}$ is measurable and then show that it lies in $L^1(\Omega)$. To show measurability, we rewrite $C_1(\omega)$ as

$$C_1(\omega) = \max\left\{\frac{2R^2}{\mu_1^2(\omega)} + \frac{2}{\mu_1(\omega)\mu_2(\omega)}\left(R + \frac{d-1}{2k_0}\right)^2, \frac{2R^2}{\mu_1(\omega)\mu_2(\omega)} + \frac{2}{\mu_2^2(\omega)}\left(R + \frac{d-1}{2k_0}\right)^2\right\}.$$

The functions μ_1^{-1} and μ_2^{-1} are measurable by assumption; to conclude C_1 is measurable we use the facts (see e.g. [28, Theorems 19.C, 20.A]): (i) the square of a measurable function is measurable, and (ii) the product, sum, and maximum of two measurable functions are measurable. Under Condition 1.3, the function f lies in the Bochner space $L^2(\Omega; L^2(D_R))$. Therefore, f is strongly measurable and hence f is measurable by Theorem B.18. The map $f \mapsto \|f\|^2_{L^2(D_R)}$ is clearly continuous, and therefore f_1 is measurable by Lemma B.4. As the product of two measurable functions is measurable, it follows that $C_1\,\|f\|^2_{L^2(D_R)}$ is measurable.

We now show that $C_1\|f\|^2_{L^2(D_R)} \in L^1(\Omega)$. The assumptions $1/\mu_1, 1/\mu_2 \in L^2(\Omega)$ and the Cauchy–Schwarz inequality imply $1/(\mu_1\mu_2) \in L^1(\Omega)$. Therefore the maps,

$$\omega \mapsto \frac{2R^2}{\mu_1^2(\omega)} + \frac{2}{\mu_1(\omega)\mu_2(\omega)}\left(R + \frac{d-1}{2k_0}\right)^2 \text{ and } \omega \mapsto \frac{2R^2}{\mu_1(\omega)\mu_2(\omega)} + \frac{2}{\mu_2^2(\omega)}\left(R + \frac{d-1}{2k_0}\right)^2$$

are in $L^1(\Omega)$. Since the maximum of two functions in $L^1(\Omega)$ is also in $L^1(\Omega)$, it follows that $C_1 \in L^1(\Omega)$. Condition 1.3 implies that $\|f\|^2_{L^2(D_R)} \in L^1(\Omega)$.

To conclude $C_1\|f\|^2_{L^2(D_R)} \in L^1(\Omega)$, observe that the only dependence of C_1 on ω is through μ_1 and μ_2. As μ_1 and μ_2 are assumed independent of f, and measurable functions of independent random variables are independent [37, p.236] it follows that C_1 and $\|f\|^2_{L^2(D_R)}$ are independent, and therefore

$$\left\|C_1\|f\|^2_{L^2(D_R)}\right\|_{L^1(\Omega)} = \int_\Omega C_1(\omega)\|f(\omega)\|^2_{L^2(D_R)}\,\mathrm{d}\mathbb{P}(\omega) = \|C_1\|_{L^1(\Omega)}\left\|\|f\|^2_{L^2(D_R)}\right\|_{L^1(\Omega)} < \infty. \tag{4.3}$$

Therefore $C_1\|f\|^2_{L^2(D)} \in L^1(\Omega)$ as required. We take the expectation (equivalently, the L^1 norm) of (4.2) (with $A_0 = A(\omega)$ etc.) and use (4.3) to obtain (1.8). ■

Remark 4.17 (The case when f, μ_1, and μ_2 are not independent). Remark 1.9 shows that for the physically relevant example of scattering by a plane wave, f, μ_1, and μ_2 may not be independent. In this case, if we replace the requirements in Condition 1.6 that $f \in L^2\big(\Omega; L^2(D)\big)$ and $1/\mu_1$, $1/\mu_2 \in L^2(\Omega)$ with the stronger requirements $f \in L^4\big(\Omega; L^2(D)\big)$ and $1/\mu_1$, $1/\mu_2 \in L^4(\Omega)$, then one can obtain the bound

$$\|\nabla u\|^2_{L^2\left(\Omega; H^1_{0,D}(D_R)\right)} + k^2\|u\|^2_{L^2\left(\Omega; H^1_{0,D}(D_R)\right)} \leq \|C_1\|_{L^2(\Omega)}\|f\|^2_{L^4(\Omega; L^2(D_R))}.$$

Indeed, instead of independence, we use the Cauchy–Schwartz inequality in (4.3) to conclude

$$\left\|C_1\|f\|^2_{L^2(D_R)}\right\|_{L^1(\Omega)} \leq \|C_1\|_{L^2(\Omega)}\left\|\|f\|^2_{L^2(D_R)}\right\|_{L^2(\Omega)} = \|C_1\|_{L^2(\Omega)}\|f\|^2_{L^4(\Omega; L^2(D_R))}.$$

Lemma 4.18 (Condition L2 for Helmholtz stochastic EDP). *If $f \in L^2\big(\Omega; L^2(D_R)\big)$ and A and n are strongly measurable, then Condition L2 holds for the Helmholtz stochastic EDP.*

Proof of Lemma 4.18. Since A, n, and f are strongly measurable, Conditions C1 and C2 hold by Lemma 4.8; i.e., c is both measurable and $\mathbb{P}$-essentially separably valued. Furthermore, by Theorem B.18 c is strongly measurable. By Lemma 4.9, Condition L1 holds, so the map $\mathcal{L}$ is continuous. Hence, by Lemma B.21, $\mathcal{L} \circ c$ is strongly measurable. We also have that $\|(\mathcal{L} \circ c)(\omega)\|_{Y^*} = \|f(\omega)\|_{L^2(D_R)}/k$, and thus $\mathcal{L} \circ c \in L^2(\Omega; Y^*)$ since $f \in L^2\big(\Omega; L^2(D_R)\big)$. ■

Lemma 4.19 (Condition A2 for the Helmholtz stochastic EDP).
If $A \in L^\infty\big(\Omega; L^\infty\big(D_R; \mathbb{R}^{d\times d}\big)\big)$, $n \in L^\infty(\Omega; L^\infty(D_R; \mathbb{R}))$, and f is strongly measurable, then Condition A2 holds for the Helmholtz stochastic EDP.

Proof of Lemma 4.19. A near-identical argument to that at the beginning of the proof of Lemma 4.18 shows $\mathcal{A} \circ c$ is strongly measurable. Recall that the Dirichlet-to-Neumann operator T_R is continuous from $H^{1/2}(\Gamma_R)$ to $H^{-1/2}(\Gamma_R)$, see e.g. [42, Theorem 2.6.4]. Let $v_1 \in X, v_2 \in Y$, and observe that the Cauchy–Schwartz inequality and these properties of T_R

imply that there exists $C(k) > 0$ such that

$$\left|\Big[\big[\mathcal{A}_{c(\omega)}\big](v_1)\Big](v_2)\right| \leq \|A(\omega)\|_{L^\infty(D_R;\mathbb{R}^{d\times d})}\|\nabla v_1\|_{L^2(D_R)}\|\nabla v_2\|_{L^2(D_R)}$$
$$+ k^2\|n(\omega)\|_{L^\infty(D_R;\mathbb{R})}\|v_1\|_{L^2(D_R)}\|v_2\|_{L^2(D_R)}$$
$$+ C(k)\|\gamma v_1\|_{H^{1/2}(\Gamma_R)}\|\gamma v_2\|_{H^{1/2}(\Gamma_R)},$$

where we have used the fact that the two norms

$$\operatorname{ess\,sup}_{\mathbf{x}\in D_R}\|A(\omega,\mathbf{x})\|_2 \quad \text{and} \quad \|A(\omega)\|_{L^\infty(D_R;\mathbb{R}^{d\times d})} := \max_{i,j\in\{1,\ldots,d\}}\|A_{i,j}(\omega)\|_{L^\infty(D_R;\mathbb{R})} \tag{4.4}$$

are equivalent. Since the trace operator γ is continuous from $H^1(D_R)$ to $H^{1/2}(\Gamma_R)$ (see, e.g. [39, Theorem 3.38]), there exists $\widetilde{C} > 0$ such that

$$\|(\mathcal{A}\circ c)(\omega)\|_{\mathrm{B}(X,Y^*)} \leq \widetilde{C}\max\Big\{\|A(\omega)\|_{L^\infty(D_R;\mathbb{R}^{d\times d})}, \|n(\omega)\|_{L^\infty(D_R;\mathbb{R})}, C(k)\Big\}\|v_1\|_{1,k}\|v_2\|_{1,k}.$$

and hence $\mathcal{A}\circ c \in L^\infty(\Omega; \mathrm{B}(X,Y^*))$. ■

4.3. Proofs of Theorems 1.4 and 1.8.

Proof of Theorem 1.4. We construct a solution of Problem 1 by letting $u = \mathcal{S}\circ c$ (which is well-defined by Theorem 4.11); by construction, $[a(\omega)](u(\omega), v) = [L(\omega)](v)$ for all $v \in H^1_{0,D}(D_R)$ almost surely. It follows that u is measurable by Condition 1.3 and Lemmas 4.12, 4.12, and B.4, and so u solves Problem 1. We therefore proceed to apply the general theory.

Conditions A1 and L1 hold by Lemma 4.9; Condition A2 holds by Lemma 4.19; Condition L2 holds by Lemma 4.18; Conditions C1 and C2 hold by Lemma 4.8 and Condition 1.3; and Condition U holds by Lemma 4.13. Therefore we can apply Theorems 2.8 and 2.9 and Lemmas 2.7 and 2.11 to conclude the results. ■

Proof of Theorem 1.8. All the conclusions of Theorem 1.4 hold, and we only need to show that if u solves Problem 1 then it also solves Problem 2. Condition B holds by Conditions 1.3 and 1.6 and Lemma 4.16. The result then follows from Theorem 2.6. ■

Appendix A. Failure of Fredholm theory for the stochastic variational formulation of Helmholtz problems.

The standard approach to proving existence and uniqueness of a (deterministic) Helmholtz BVP is to show that the associated sesquilinear form satisfies a Gårding inequality, and then apply Fredholm theory to deduce that existence and uniqueness are equivalent; see, e.g., [39, Theorem 4.10]. This procedure relies on the fact that the inclusion $H^1_{0,D}(D_R) \hookrightarrow L^2(D_R)$ is compact; see, e.g., [39, Theorem 3.27].

As noted in subsection 1.3, the analysis in [18] of Problem 3 for the Helmholtz Interior Impedance Problem mimics this approach and assums that $L^2\big(\Omega; H^1(D)\big)$ is compactly contained in $L^2\big(\Omega; L^2(D)\big)$, where D is the spatial domain. Here we briefly show $L^2\big(\Omega; H^1(D)\big)$ is *not* compactly contained in $L^2\big(\Omega; L^2(D)\big)$ by giving an explicit example of a bounded sequence in $L^2\big(\Omega; H^1(D)\big)$ that has no convergent subsequence in $L^2\big(\Omega; L^2(D)\big)$. Necessary and sufficient conditions for a subset of $L^p([0,T]; B)$, for B a Banach space, to be compact, can be found in [49]. In particular, [49] shows that a space C being compactly contained in a space B does not by itself imply $L^2([0,T]; C)$ is compactly contained in $L^2([0,T]; B)$.

Example A.1. *Let* $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \lambda)$. *Let* D *be a compact subset of* $\mathbb{R}^d$. *Since* $L^2(\Omega)$ *is separable, it has an orthonormal basis, which we denote by* $(f_m)_{m\in\mathbb{N}}$. *Let* $u_m \in L^2\big(\Omega; H^1(D)\big)$ *be defined by* $u_m(\omega)(x) := f_m(\omega)$, *for all* $x \in D$, *i.e., for each value of* ω, $u_m(\omega)$ *is a constant function on* D *and so* $\|u_m(\omega)\|_{H^1(D)} = \|u_m(\omega)\|_{L^2(D)}$. *Then*

$$\|u_m\|^2_{L^2(\Omega;H^1(D))} = \int_\Omega \|u_m(\omega)\|^2_{H^1(D)}\,\mathrm{d}\mathbb{P}(\omega) = \lambda(D)^2 \int_\Omega |f_m(\omega)|^2\,\mathrm{d}\mathbb{P}(\omega) = \|f_m\|^2_{L^2(\Omega)}\lambda(D)^2,$$

and so u_m *is a bounded sequence in* $L^2\big(\Omega; H^1(D)\big)$. *However, for* $n \neq m$, *we have*

$$\|u_m - u_n\|^2_{L^2(\Omega;L^2(D))} = \lambda(D)^2 \int_\Omega |u_m(\omega) - u_n(\omega)|^2\,\mathrm{d}\mathbb{P}(\omega) = \lambda(D)^2\|f_m - f_n\|^2_{L^2(\Omega)} = 2\lambda(D)^2$$

if $n \neq m$, *since the* f_m *form an orthonormal basis for* $L^2(D)$. *Therefore* $(u_m)_{m\in\mathbb{N}}$ *is bounded in* $L^2\big(\Omega; H^1(D)\big)$ *but does not have a convergent subsequence in* $L^2\big(\Omega; L^2(D)\big)$, *and thus the inclusion of* $L^2\big(\Omega; H^1(D)\big)$ *into* $L^2\big(\Omega; L^2(D)\big)$ *cannot be compact.*

Appendix B. Recap of basic material on measure theory and Bochner spaces. We include this section, not only for completeness, but also to aid readers of this paper who are more familiar with deterministic, as opposed to stochastic, Helmholtz problems. Recall that here, and in the rest of the paper, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space.

B.1. Recap of measure theory results. We first recall some results from measure theory, with our main reference [7]. Even though [7] mainly considers maps with image $\mathbb{R}$, the results we quote for more general images are straightforward generalisations of the results in [7].

Definition B.1 (Measurable map). *If* $(M, \mathcal{M})$ *and* $(N, \mathcal{N})$ *are measurable spaces, we say that* $f : M \to N$ *is measurable (with respect to* $(\mathcal{M}, \mathcal{N})$*) if* $f^{-1}(E) \in \mathcal{M}$ *for all* $E \in \mathcal{N}$.

Definition B.2 (Borel σ-algebra). *If* $(S, \mathcal{T}_S)$ *is a topological space, the* Borel σ-algebra $\mathcal{B}(S)$ *on* S *is the* σ*-algebra generated by* $\mathcal{T}_S$.

If V is any topological space (including a Hilbert, Banach, metric, or normed vector space) then we will take always the Borel σ-algebra on V unless stated otherwise.

Lemma B.3 (Continuous maps are measurable [7, Theorem 2.1.2]). *Any continuous function between two topological spaces is measurable.*

Lemma B.4 (The composition of a measurable and a continuous map is measurable [7, p. 146]). *Let* $(M, \mathcal{M})$ *be a measurable space and let* $(S, \mathcal{T}_S)$ *and* $(T, \mathcal{T}_T)$ *be topological spaces. Let* $f : M \to S$ *be measurable and let* $h : S \to T$ *be continuous. Then* $h \circ f$ *is measurable.*

Definition B.5 (Product σ-algebra [17, Section IV.11]). *Let* $(M_1, \mathcal{M}_1), \ldots, (M_m, \mathcal{M}_m)$ *be measurable spaces. The* product σ-algebra $M_1 \otimes \cdots \otimes M_m$ *is defined as the* σ*-algebra generated by the set of measurable rectangles* $\{R_1 \times \cdots \times R_m : R_1 \in \mathcal{M}_1, \ldots, R_m \in \mathcal{M}_m\}$.

Lemma B.6 (Measurability of the Cartesian product of measurable functions).
Let $(M_1, \mathcal{M}_1), \ldots, (M_m, \mathcal{M}_m)$ *be measurable spaces and* $h_j : \Omega \to M_j$, $j = 1, \ldots, m$ *be measurable functions. Then the product map* $P : \Omega \to M_1 \times \cdots \times M_m$ *given by* $P(\omega) := (h_1(\omega), \ldots, h_m(\omega))$ *is measurable with respect to* $(\mathcal{F}, \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_m)$.

Sketch proof of Lemma B.6. Let $\mathrm{Rect}(\mathcal{M}_1,\ldots,\mathcal{M}_m)$ denote the set of measurable rectangles, as in Definition B.5. Let $\mathcal{P} := \left\{C \subseteq M_1\times\cdots\times M_m : P^{-1}(C)\in\mathcal{F}\right\}$. The proof of the lemma consists of the following straightforward steps, whose proofs are omitted: (i) Show $\mathrm{Rect}(\mathcal{M}_1,\ldots,\mathcal{M}_m)\subseteq\mathcal{P}$. (ii) Show $\mathcal{P}$ is a σ-algebra. (iii) Deduce $\mathcal{M}_1\otimes\cdots\otimes\mathcal{M}_m\subseteq\mathcal{P}$ (since $\mathcal{M}_1\otimes\cdots\otimes\mathcal{M}_m$ is generated by measurable rectangles). (iv) Conclude P is measurable with respect to $(\mathcal{F},\mathcal{M}_1\otimes\cdots\otimes\mathcal{M}_m)$. ∎

Lemma B.7 (Product of Borel σ-algebras is Borel σ-algebra of the product [7, Lemma 6.2.1 (i)]). *Let H_1, H_2 be Hausdorff spaces and let H_2 have a countable base (e.g. H_2 could be a separable metric space). Then $\mathcal{B}(H_1\times H_2)=\mathcal{B}(H_1)\otimes\mathcal{B}(H_2)$, where $\mathcal{B}(H_1\times H_2)$ is the Borel σ-algebra of the product topology on $H_1\times H_2$.*

B.2. Recap of results on Bochner spaces. We now recap the theory of Bochner spaces, using [16] as our main reference. In what follows the space V is always a Banach space.

Definition B.8 (Simple function). *A function $v:\Omega\to V$ is* simple *if there exist $v_1,\ldots,v_m\in V$ and $E_1,\ldots,E_m\in\mathcal{F}$ such that $v=\sum_{i=1}^m v_i\chi_{E_i}$, where χ_{E_i} is the indicator function on E_i.*

Definition B.9 (Strongly measurable). *A function $v:\Omega\to V$ is* strongly measurable [1] *if there exists a sequence of simple functions $(v_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\|v_n-v\|_V=0$, $\mathbb{P}$-almost everywhere.*

Definition B.10 (Bochner integrable [16, p. 49]). *A strongly measurable function $v:\Omega\to V$ is called* Bochner integrable *if there exists a sequence of simple functions $(v_n)_{n\in\mathbb{N}}$ such that* $\lim_{n\to\infty}\int_\Omega\|v_n(\omega)-v(\omega)\|_V\,\mathrm{d}\mathbb{P}(\omega)=0$.

Theorem B.11 (Condition for Bochner integrability [16, Theorem II.2.2]). *A strongly measurable function $v:\Omega\to V$ is Bochner integrable if and only if $\int_\Omega\|v\|_V\,\mathrm{d}\mathbb{P}<\infty$.*

Corollary B.12 (Sufficient condition for Bochner integrability). *Let $p\geq 1$. If a strongly measurable function $v:\Omega\to V$ has $\int_\Omega\|v\|_V^p\,\mathrm{d}\mathbb{P}<\infty$, then v is Bochner integrable.*

Definition B.13 (Bochner norm). *For a Bochner integrable function $v:\Omega\to V$, let*

$$\|v\|_{L^p(\Omega;V)} := \left(\int_\Omega\|v(\omega)\|_V^p\,\mathrm{d}\mathbb{P}(\omega)\right)^{1/p},\ 1\leq p<\infty,\ \text{and } \|v\|_{L^\infty(\Omega;V)} := \operatorname{ess\,sup}_{\omega\in\Omega}\|v(\omega)\|_V.$$

Definition B.14 (Bochner space). *Let $1\leq p\leq\infty$. Then*

$$L^p(\Omega;V) := \left\{v:\Omega\to V : v \text{ is Bochner integrable}, \|v\|_{L^p(\Omega;V)}<\infty\right\}.$$

Definition B.15 (Complete probability space). *A probability space $(\Omega,\mathcal{F},\mathbb{P})$ is complete if for every $E_1\in\mathcal{F}$ with $\mathbb{P}(E_1)=0$, the inclusion $E_2\subseteq E_1$ implies that $E_2\in\mathcal{F}$.*

Definition B.16 (Separable space). *A topological space is* separable *if it contains a countable, dense subset.*

Definition B.17 (σ-finite). *A probability space $(\Omega,\mathcal{F},\mathbb{P})$ is* σ-finite *if there exist $E_1,E_2,\ldots\in\mathcal{F}$ with $\mathbb{P}(E_m)<\infty$ for all $m\in\mathbb{N}$ such that $\Omega=\cup_{m=1}^\infty E_m$.*

[1] In [16] the authors use the term *μ-measurable* instead of *strongly measurable* (where μ is the measure on the domain of the functions under consideration).

Theorem B.18 (Pettis measurability theorem [47, Proposition 2.15]). *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete* σ*-finite measure space. The following are equivalent for a function* $v : \Omega \to V$*: (i)* v *is strongly measurable, (ii)* v *is measurable and* $\mathbb{P}$*-essentially separably valued.*

Corollary B.19 (Equivalence of measurable and strongly measurable when the image is separable). *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a* σ*-finite measure space. If* V *is a separable Banach space, then a function* $v : \Omega \to V$ *is strongly measurable if, and only if, it is measurable.*

Lemma B.20 (The composition of a continuous map and a $\mathbb{P}$-essentially separably valued map). *Let* $(S, \mathcal{T}_S)$ *and* $(T, \mathcal{T}_T)$ *be topological spaces. If* $f_1 : \Omega \to S$ *and* $f_2 : S \to T$ *are such that* f_1 *is* $\mathbb{P}$*-essentially separably valued and* f_2 *is continuous, then* $f_2 \circ f_1$ *is* $\mathbb{P}$*-essentially separably valued.*

Proof of Lemma B.20. As f_1 is $\mathbb{P}$-essentially separably valued, there exists $E \in \mathcal{F}$ such that $\mathbb{P}(E) = 1$ and $f_1(E) \subseteq G \subseteq S$, where G is separable. As f_2 is continuous, $f_2(G)$ is separable [53, Theorem 16.4(a)]. Therefore, since $(f_2 \circ f_1)(E) \subseteq f_2(G)$, it follows that $f_2 \circ f_1$ is $\mathbb{P}$-essentially separably valued. ∎

Lemma B.21 (The composition of a continuous map and a strongly measurable map). *If* B_1 *and* B_2 *are Banach spaces and there exist* $f_1 : \Omega \to B_1$ *and* $f_2 : B_1 \to B_2$ *such that* f_1 *is strongly measurable and* f_2 *is continuous, then* $f_2 \circ f_1$ *is strongly measurable.*

Proof of Lemma B.21. By Theorem B.18, f_1 is both measurable and $\mathbb{P}$-essentially separably valued. We then apply Lemmas B.4 and B.20 to conclude $f_2 \circ f_1$ is both measurable and $\mathbb{P}$-essentially separably valued. Hence by Theorem B.18 $f_2 \circ f_1$ is strongly measurable. ∎

Lemma B.22 (Zero in all integrals implies zero almost everywhere [16, Corollary II.2.5]). *If* α *is Bochner integrable and* $\int_E \alpha(\omega)\, \mathrm{d}\mathbb{P}(\omega) = 0$ *for each* $E \in \mathcal{F}$ *then* $\alpha = 0$ $\mathbb{P}$*-almost everywhere.*

Lemma B.23 (Cartesian product of $\mathbb{P}$-essentially separably valued maps). *Let* $(\mathcal{C}_1, \mathcal{T}_{\mathcal{C}_1}), \ldots, (\mathcal{C}_m, \mathcal{T}_{\mathcal{C}_m})$ *be topological spaces, and let* $s_j : \Omega \to \mathcal{C}_j$*,* $j = 1, \ldots, m$ *be* $\mathbb{P}$*-essentially separably valued. Define* $\mathcal{C} := \mathcal{C}_1 \times \cdots \times \mathcal{C}_m$ *and equip* $\mathcal{C}$ *with the product topology. Then the map* $f : \Omega \to \mathcal{C}$ *given by* $s(\omega) := (s_1(\omega), \ldots, s_m(\omega))$ *is* $\mathbb{P}$*-essentially separably valued.*

The proof of Lemma B.23 is straightforward and omitted.

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